1-1-1982

The use of letters in precalculus algebra.

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THE USE OF LETTERS IN PRECALCULUS ALGEBRA

A Dissertation Presented

By

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Submitted to the Graduate School of the
University of Massachusetts in partial fulfillment
of the requirements for the degree of

DOCTOR OF EDUCATION

May 1982

Education

Research reported in this paper was supported by NSF awards nos. SED78-22043, SED80-16567, and SED80-20020 in the Joint National Institute of Education- National Science Foundation Program of Research on Cognitive Processes and the Structure of Knowledge in Science and Mathematics.

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ACKNOWLEDGMENT

I would like to express my thanks and appreciation to the many people who have given me valuable help and support in the evolution and development of this work. John Clement, my immediate advisor, has spent many long hours of brainstorming with me and many hours of reading, critiquing, and supporting my efforts. It is his extensive pioneering work in this area that has provided a foundation for my work. In addition to John Clement, I am grateful to all of the rest of the people at the Cognitive Development Project of the Department of Physics and Astronomy, directed by John Lochhead, for their support, both intellectual, and technical.

William Masalski, the chair of my committee has advised me well from the beginning of my graduate school career and has continued to do so throughout the dissertation process. He has gently, patiently but persistently helped me to keep my work on course. I am also grateful to the other members of my committee, Richard Konicek and John Lochhead for their helpful guidance. In addition, the conversations I have had with James Kaput have inspired many of the directions of this work and I thank him for that.

Finally, I would like to thank Sandra, Michael and Tom whose love, friendship and counsel have sustained me through this project.

DEDICATION

I dedicate this dissertation, with all of my love, to Sandra.

ABSTRACT

THE USE OF LETTERS IN PRECALCULUS ALGEBRA

(May, 1982)

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The use of letters in precalculus algebra is investigated from three perspectives: what letters mean, how letters are presented, and how students view letters while solving word problems.

Five characteristics of letters are identified as critical indicators of letter use and meaning: domain, dimension, variability, solution set, and semantic meaning. Letters with semantic meaning are defined as _semantically laden letters_.

Forty-one secondary and college mathematics texts are analyzed for how these five characteristics are presented. Special attention is paid to the texts' treatment of semantically laden letters. Tables are given which indicate, among other things, the diversity of presentations and the overall paucity of problems that require some interpretation of the letters' semantic meaning.

Diagnostic tests and clinical interviews of college students were conducted to test an hypothesis that students view semantically laden letters as labels for concrete

entities. This hypothesis proved to be inadequate because students not only view letters qualitatively as labels, but also attribute quantitative value to them and do so with more than one attribute.

A second phase of research consisted of more clinical interviews, the microanalysis of the interview transcripts, and additional diagnostic testing. It was found that most students view semantically laden letters as undifferentiated conglomerates, i.e., as having global, amorphous, undifferentiated meaning.

Four behaviors indicative of the above conception are identified and described. Transcripts from clinical interviews are used extensively to illustrate these four behaviors. Eight of nine students exhibited three of the four behaviors, strongly indicating that they view semantically laden letters as undifferentiated conglomerates.

The data from two diagnostic tests are presented. They indicate that college math students fare poorly at solving relatively simple algebra word problems, and that there is a positive correlation between evidence of the four behaviors and poorer scores on the tests.

Students' tendency to only vaguely and amorphously define the letters they use is shown to have implications not only for mathematics but also for cognitive science.

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INTRODUCTION

As early as 1956, Karl Menger, a research mathematician at the Illinois Institute of Technology, lamented the state of students' understanding of the use of literals in algebra.

> On blackboards, they see numerous expressions and formulas including letters, especially x and y; but they have to discern exactly how these letters are used. In lectures, they hear a great deal about variables and constants; but they have to surmise what those terms mean...Is it surprising that many a beginner gives up? Can we blame those who halt at the stage where habitual repetition of words becomes a substitute for true understanding of procedures? (p.iii)

Menger noted that a plethora of diverse uses of literals in algebra were all categorized under umbrella terms such as variable or constant. Finer distinctions, however, were tacit ones apparent only to particularly clever students and active mathematicians, the latter being referred to as "virtuosos."

> Becoming an active mathematician for the past 200 years has meant, and to this day means, becoming familiar with the antiquated conceptual and symbolic frame of Renaissance mathematicians to the point of applying the various types of variables efficiently and of manipulating the letters x and y in their discrepant uses with virtuosity. (p.iv)

Menger wrote in the pre-Sputnik year of 1956. The concerns of the day were the "eminent...shortages of active mathematicians and... mathematically trained scientists." (p.iii) Menger feared that "even keen logicians [would be] deterred" from the study of mathematics because where they anticipated "pure reasoning," they found a "need for mere guessing." (p.v)

Menger proposed the following three steps "on the way to a solution of the problem":

> 1) The basic concepts of mathematics must be analyzed; fundamental terms, such as variable, constant, parameter, indeterminate, and the like must be freed of their current equivocations and resolved in the full spectra of their meanings. The corresponding procedures and manipulations of symbols must be clearly distinguished and separated from each other.
> 2) The vocabulary of the analyzed concepts and symbols as well as the grammar of the clarified procedures must be developed as the epitome of common sense (which they really are) and not as either esoteric magic or incomprehensible abstraction (as which, even to some quite intelligent youngsters, traditional mathematics appears).
> 3) The system thus obtained must be presented to students; their reactions must be observed dispassionately and recorded in a scientific spirit. For, under the circumstances, it is the students who eventually will decide the form of the mathematics...(p.iv)

The problem, as Menger described it in 1956, is strongly evident today. The use of literals in algebra is still ill-defined; many of the concepts are still tacit; and many students reveal confused conceptions or troublesome misconceptions concerning letter use in algebra.

There is, however, one significant and crucial difference between the problem in 1956 and the problem today. Menger was most concerned about those "quite intelligent youngsters" and "keen logicians" for whom the symbols in mathematics appeared too haphazard and "magical." The presentation of symbols, according to Menger, did not meet the "standards of logical clarity" for those students. (p.iv)

Today's mathematics classes, however, are increasingly comprised of more than budding mathematicians, logicians, and scientists. More and more professions are requiring a certain literacy in higher and higher levels of mathematics. As a result, many students in today's math classes are concerned with issues far more elementary than the logical rigor and preciseness of the material presented. There are some researchers who claim that a significant proportion of adults and people in their late teens (including those enrolled in college math classes) have not reached Piaget's level of formal reasoning (Fuller, Karplus, and Lawson (1977); Renner (1976); Collis (1975)). Such students would not be troubled with some logical inconsistency in an abstract system but with the very act of using an abstract system to symbolize concrete entities. Thus, letters used in the solution of a word problem, being abstract symbols with concrete referents, pose a special problem to today's math students. Letters must not only be analyzed and understood as part of an abstract system, but must also be understood when couched in the semantic context of word problems.

The need for an assessment of student understanding of algebraic letters exists today as it did in 1956, though the forms of that assessment must be significantly altered. It is no longer sufficient to say that students' problems concerning letters in algebra result primarily from the lack of logical clarity in their presentation. It is no longer sufficient to assume that all that needs to be done is to refine the abstract symbol system, making it more rigorous and pure. The meaning of abstract symbols must be reassessed, taking into consideration their concrete referents and the context in which they are found. How these symbols are presented in textbooks must also be analyzed and reviewed in the same terms. Most importantly, student conceptions and misconceptions must be understood anew.

Matter Meant, Matter Taught, Matter Learned

Menger's three-step plan toward "solving" the problem of misunderstood letter use in algebra suggests a three-tiered approach to studying letter use today.

1. How are letters used in algebra.
2. How is letter use taught.
3. What are some student conceptions of how letters are used.

This three sided approach to looking at a problem is consistent with the ideas of Bauersfeld (1976). Herscovics (1979) cites Bauersfeld on distinguishing between "matter meant," "matter taught," and "matter learned." (p.93) In education in general, that which is taught is not always that which was meant. That which the students have learned is certainly not always that which was taught.

This dissertation will explore those three facets of algebra. Throughout the remainder of this dissertation, the word "matter" will refer to the use of letters in algebra. Chapter II, entitled "Matter Meant," will attempt to build a mechanism for analyzing letter use in algebra problems. This mechanism or tool could be used in distinguishing variations in letter use from one problem to another. It underscores the salient characteristics of a particular letter (vis-a-vis the context in which it is found). Furthermore, it takes into consideration the added semantic content that comes from the context of a word problem.

What is "meant" by a letter in a particular context can be thought of as a benchmark against which one can compare what has been taught and what has been learned. To that end, the mechanism by which one understands matter meant should provide a framework for analyzing matter taught and matter learned. Chapter III, entitled "Matter Taught," reviews algebra textbooks of various levels using the framework developed in Chapter II.

Chapters IV and V deal with "matter learned," i.e., student conceptions of letters in algebra. The focus, however, is limited to student conceptions of those letters that have added semantic meaning. Letters with added semantic meaning are those that are usually used in the context of solving word problems and mean more than a number. They mean a number or amount of some thing. These letters help to form a bridge between meaning in language and meaning in mathematics, and for that reason, are crucial for the application of algebra and higher level mathematics to the real world.

Chapter V describes and documents with clinical interviews a tendency of college math students to overly associate a letter with a complex referent; that is, for letters that mean "a number or amount of some thing," the students' focus is often more on the "thing" than on "number or amount." Nevertheless, the student does at times attribute quantitative meaning to the letter. The result is a mixture of disparate meanings; a vague, amorphous, unstable conception of what letters mean. This congeries of meaning is described in Chapter V as an <u>undifferentiated conglomerate</u>.

Chapter VI discusses some of the implications of the previous chapters, focusing on directions that future research could follow. It is broken into four sections, the first three of which reflect on information presented in Chapters II, III, and IV and V, respectively. The fourth section discusses some issues of cognitive science that the concept of an "undifferentiated conglomerate" uncovers. Chapter VII summarizes the entire dissertation.

Chapter I of this dissertation, which follows, will present some of the relevant research related to the topic of student conceptions of letters in algebra.

CHAPTER I

RESEARCH RELATED TO STUDENTS' UNDERSTANDING OF THE CONCEPT OF VARIABLE

The premise by which this work is justified is that students have misconceptions concerning and are confused about the use of letters in algebra. This premise is based on the results of a body of research that is described below.

Much work has been done at the Cognitive Development Project at the University of Massachusetts and by James Kaput and others involving the testing and interviewing of students doing problems that require them to translate from one symbol system to another. (See Clement, 1982; Clement, Lochhead and Monk, 1981; and Kaput, 1981, for a more extensive description of some of this work.) In some tasks, students were asked to translate an English sentence to an algebraic equation, or vice versa. In others, students were asked to interpret information in tabular, graphic, or pictorial form into an algebraic equation. The results of these studies were that many students fare poorly at these translation activities. The following two problems were among those given on diagnostic tests:

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

> Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel." Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered.

In a group of 150 first-year Engineering majors, only 63% answered the Students and Professors problem correctly, and only 27% answered the Cheesecake problem correctly. (See Kaput and Clement, 1979; and Clement, Lochhead, and Monk, 1981.) There was a very strong pattern in the errors on these problems: two thirds of the errors in both cases took the form of a <u>reversed</u> <u>equation</u>, such as 6S=P or 4C=5S. Further results indicate that students in the social sciences do considerably worse on these questions, as would be expected. (In preliminary tests, only 43% of these students answered the Students and Professors problem correctly.)

Initially, it was thought that these mistakes were simply careless misinterpretations due to specific wording of the problem. However, the reversal was found to be quite common in problems which called for translations from pictures to equations or data tables to equations. This suggested that the reversal error was not primarily due to the specific wording used in a word problem. In addition, lengthy video-taped interviews with students indicated that the difficulty was quite persistent in many cases.

Many students appeared to use a word order matching strategy by simply writing down the symbols 6S=P in the same order as the corresponding words in the text. Others, however, demonstrated a general semantic understanding of the problem (i.e., that there are, in fact, more students than professors), yet they persisted in writing reversed equations. The interviews revealed disturbing difficulties in the students' conceptualization of the basic ideas of equation and variable. For example, some showed what was called a figurative concept of equation, i.e., that since there are more students than professors, the coefficient, 6, by virtue of the fact that it is bigger than 1, should be associated with the "bigger variable," S. This resulted in the reversed equation, 6S=P. (See Clement, 1980.)

Other students had explicitly demonstrated erroneous and/or unstable conceptions of variable. Many of the college students that had been interviewed and tested acted as though they did not recognize the use of letters as standing for numbers. They seemed to confuse the use of letter as a variable with the use of letter as a label or unit. These students also tended to write the reversed equation 6S=P as the answer to the Students and Professors problem. When questioned, they read the equation as "six students for every professor" and directly identified the letter S as a label standing for "students" rather than the proper, "number of students."

Rosnick and Clement (1980) have shown that these misconceptions about variables and equations appear to be deep seated and resistant to change. In two separate series of clinical interviews of students who initially reversed the students and professors problem, students were tutored extensively, to minimal avail. Several different tutoring strategies were tried and, though the interviewer was for the most part able to change the students' behavior (i.e., they became able to write the correct equations), continued probing indicated that the students still held on to their original misconceptions. This Rosnick, Clement paper, considered part of the preliminary research for this dissertation, is given in its entirety as Appendix A.

The tendency to associate a letter with a label rather than a variable is an instance of what Kaput (1980) refers to as a "nominalist" error.(p.3) Kaput believes that this type of error is not restricted to the types of problems described above but describes a prevalent attitude of students in dealing with variables in many contexts.

Galvin and Bell (1977) coined the phrase "fruit salad algebra" (p.24) to refer to the identification of a letter as a label. In fruit salad algebra, "a" stands for apples rather than the number of apples. They provide extensive clinical interview data that demonstrates children's tendency to identify letters as labels.

Matz is another researcher who has noted the tendency to think of a letter as standing for a label.(Matz, 1979)

In her paper, she is concerned with the conceptual transition students must make from arithmetic to algebra. She identifies variables, along with equality and "non-algorithmic styles of reasoning" (p.14) as being a major conceptual step in that transition. She believes that some student errors in interpreting variables result from an incomplete transition away from arithmetic. For example, in arithmetic, the concatenation of two digits means something different than the concatenation of two letters or one letter and a digit in algebra. If a student has not grasped that distinction, they might be inclined to conclude that if 4x=46, then x=6.

Matz believes that conceiving of letters as labels rather than variables is another example of a conception inappropriately carried over from arithmetic. She says,

> literals function as unit labels when they occasionally appear in arithmetic problems. As unit labels they are carried along through a sequence of arithmetic operations or tagged back on after completing the appropriate arithmetic. Although more evidence is required, I believe that initially students may be misconstruing symbolic values as abstract labels. (p.9)

Like Menger, Matz has noted the difficulties that can arrise from blurring distinctions between constants, parameters, unknowns and variables. She identifies those differences in terms of how the variables vary. She says, "once recognized, conceptual subtleties like these seem transparent, and thus insignificant. But for naive students, they are significant and they accumulate, making algebra progressively more formidable." (p.12)

Wagner (1977) has noted that many students are incapable of "conserving equation" through a change of variable. Only 50% of the twelve year old students who were asked which was larger, w or n if 7xw + 22 = 109 and 7xn + 22 = 109, answered the question correctly.(p.21) Davis (1975), in analyzing the clinical interview of a twelve-year old solving an algebra problem, came to the conclusion that the student "...was not recognizing that x was, in fact, some number." (p.22)

Both Collis (1978) and Küchemann (1978) have extensively investigated children's conceptions of letters in algebra. Though their theoretical constructs will be discussed in Chapter II, it is fitting at this point to cite some of their results. Collis believes that many children cannot "accept the lack of closure" and that being able to do so comes under Piaget's formal operational stage of cognitive development.(pp.223-24) These students are uncomfortable with nonnumerical answers to algebra problems like n+5. Küchemann, testing 1000 fifteen-year-old students has obtained results supporting Collis' conclusions as well as the conclusions of many of the aforementioned researchers.

For example, 26% of the 1000 students gave the answer, 12, to the question, "If e + f = 8, then e + f = ?", rather than the "unclosed" answer, 8 + g. (Only 41% of the students gave the correct answer.) In a result that supports Wagner's finding, 75% of the students were unable to correctly answer whether the statement $L + M + N = L + P + N$ is always, sometimes, or never true.(p.24) Another question given was the following:

> Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it costs me 90 pence. If b is the number of blue pencils bought, and if r is the number of red pencils bought, what can you write down about b or r. (p.25)

Only 11% were able to answer this question correctly. A common wrong answer was $b + r = 90$. Küchemann believes that the students' interpretation of that equation would be "blue pencils plus red pencils cost 90 pence," (p.25) an answer that is consistent with Galvin and Bell's fruit salad algebra interpretation. Küchemann also has data that indicates that children often ignore variables. 31% of the students said that $n + 5$ multiplied by 4 is $n + 20$. 20% said that 4 added onto $n + 5$ is 9. (p.25)

In another article, Küchemann (1981) describes extensively the use of letters as objects and gives many examples of text books that foster that kind of association. He also notes that when letters stand for what he calls the "quality" of an object like length or cost, the difficulty exists but may be hidden. An expression like $3x + y$ which should be read as "3 bars at x pence and one packet at y pence" might be understood by the student to mean simply "3 bars and one packet." (p.2)

Herscovics and Kieran (1980), cite Wagner, Collis, Küchemann, Davis and others in support of the fact that students have misconceptions concerning the use of letters in algebra. Like Matz, they believe the problem has to do with the transition from arithmetic to algebra and have written about their constructivist teaching experiments designed to bridge the gap between the two subjects. (pp.575-76)

With the important exception of the work done by Clement, et al at the Cognitive Development Project and by Kaput, all of the aforementioned studies have focused on children's conceptions of letter use in algebra. However, the work of Clement, Lochhead, Kaput, et al, including the preliminary findings of this writer indicate that the problem extends well into the college years for many students. Tonnessen (1980), whose work will be discussed in more detail in Chapter II, has also concluded that college students have a "low attainment" of the concept of variable. (p.215)

If, as seems to be the case, college students hold many serious misconceptions concerning the use of letters in algebra (and consequently calculus), what kind of learning can they expect to achieve in their college math classes? If those misconceptions remain unaddressed, what implications does that hold for the future of both college and high school math curricula?

The use of letter symbolically is a mainstay of all of mathematics. As Menger has said, letters must be "resolved into the full spectra of their meanings," (p.iv) the full spectra of how they are taught and the full spectra of how, students understand them. This dissertation is one attempt at achieving that goal.

CHAPTER II

MATTER MEANT: THE MEANING OF LETTERS IN ALGEBRA

In this chapter, I will attempt to develop a mechanism with which one can assess what is meant by a letter in a particular algebraic context. Before doing so, the works of Tonnessen (1980), Collis (1975), and Küchemann (1978 and 1981) that pertain to "matter meant" will be presented and criticized.

What is meant by letters in algebra is open to different interpretations and can be approached in significantly different ways. One of those ways, as exemplified by the work of Tonnessen, is what can be called a general definitive approach. Tonnessen generalizes the different uses of letters under the umbrella term, <u>variable</u>, and approaches matter meant through the definition of <u>variable</u>. (For Tonnessen, the word <u>variable</u> underlined refers to the term "concept of variable.")

Another approach to "matter meant" has been taken by Küchemann. This approach could be described as a specific categorical approach. Küchemann analyzes and categorizes different specific uses of letters and makes no attempt at a unifying generalization. Whereas Tonnessen has attempted to understand the use of letters in algebra via a mathematically rigorous definition, Küchemann has done so through a qualitative analysis of different characteristic examples.

Tonnessen's General Definitive Approach

Tonnessen's dissertation is towards a doctoral degree in mathematics. As a mathematician, he seems to have recognized the importance of logical rigor and preciseness to higher mathematics.

Each axiomatic system in mathematics is founded deeply in and is dependent on the definitions and axioms from which that system has evolved. For that reason, great care must be taken in creating those definitions and axioms. In particular, definitions must not only apply to common and well known situations, but they must also be able to speak to anomalies and idiosyncrasies. In the words of Lakatos (1976), they must be able to "bar the monsters." (p.23)

Lakatos has shown in Proofs and Refutations how highly dependent the proof of a theorem is on the definition of the entities involved. He follows the development of the "proof" of Euclid,s conjecture which posited the relationship between the number of faces, edges and vertices of a polygonal figure. Someone would "prove" the conjecture whereupon someone else would discover a "monster"; that is, a type of polygon that had not been considered before. At that point, the definition would be altered, made more rigorous and precise, so as to "bar the monsters."

In addition to rigor and preciseness, a mathematical definition must also possess the attribute of being functional. That is, the definition must not be so verbally cumbersome as to preclude its mathematical usage. Furthermore, it must be expressed in a language that is consistent with that of the axiomatic system in which it is found.

An example is the ε, δ definition of continuity. Consider the following version:

> A function f(x) defined for $x = x_0$ is called continuous at x_0 if given a positive ε, we can find a positive δ such that, for all x with $[x - x_0] < \delta$, we have $[f(x_0) - f(x)] < \varepsilon$. (Lipman and Bers, 1969, p.110)

This is a definition that not only applies to familiar functions in the familiar Cartesian plane but can also be easily extended and adapted to idiosyncratic situations. Furthermore, it is expressed in a language that allows for algebraic manipulation.

Tonnessen seems to have applied similar standards in developing a definition for the concept variable. He has attempted to find a definition of the concept that is all encompassing, that is, that would apply in all of the disparate situations in which letters are used to stand for elements in a set. To that end, Tonnessen analyzes the historical development of the definition of the concept variable. He uses, in part, the work of Hamley (1934). His conclusion, based also on a review of current undergraduate math texts, is that "a valid definition [of variable] must...incorporate both <u>symbol</u> and <u>domain</u>. <u>Symbol</u> and <u>domain</u> are connected by the relational concept <u>represents any element of</u>." He defines <u>variable</u> as follows:

> <u>Definition</u>. A <u>variable</u> is an ordered pair (x, D(x)) where x is a symbol and D(x) is a set with at least two elements, such that x represents any of the elements of the set D(x). The set, D(x), is called the <u>domain</u> of x, and elements of D(x) are called the <u>values</u> for x. (pp.9-10)

Like the definition for continuity, Tonnessen's definition for variable is written in a familiar mathematical language. He points out that defining a concept in terms of an ordered pair has precedence in mathematics. For example, a topological space is defined in terms of an ordered pair (x, T(x)). Furthermore, the scope of Tonnessen's definition, like that of the definition of continuity, is broad. Whether letters are used in one "variable" solvable equations, or functional relationships, or abstract discourses, Tonnessen's definition most likely would apply.

It is interesting to note that the scope of the definition could have been even broader. If D(x) were allowed to be a one element set, then such symbols as e or π could be considered to be variables, which is the case in some algebra texts. Beberman (1960) defines a "variable quantity" in much the same way as Tonnessen has defined <u>variable</u> except that he allows for a single element domain.

In so doing, the number, 2, can be thought of as a variable quantity. (p.104)

Having definitions that are broad in scope and, at the same time, logically rigorous and precise is often possible only at a cost. One must often forsake the intuitive or qualitative meaning of that which is being defined. Though the definition may make thorough logical sense, it may do so at the expense of the reader's cognitive conceptual understanding. In the final analysis, "matter meant" might be lost in the intuitive tangle of mathematical simplicity.

Consider the concept of continuity at the level of the beginning calculus student. Intuitively, a continuous function could be described as a function that can be drawn on a graph without lifting the pencil. Innumerably many calculus students, however, are lost in the maze of ε's and δ's and because of that, may be lost to this more intuitive, albeit crude understanding. Thus the mathematical definition of continuity is not necessarily where one should go to gain understanding of what is meant by continuity.

The same may hold true for the definition of variable. First of all, matter meant may get lost in confusing symbolism. If, for Tonnessen, (x,D(x)) is a variable and x is a symbol, what is D(x)? D(x) itself is a symbol that stands for an element in the set of all possible (two elements or greater) sets. Thus one could imagine a variable (D(x), D(D(x))) where D(D(x)) is the domain of D(x).

This argument is a recursive one, and is also somewhat petty. It is not meant to imply that Tonnessen's definition of variable is fallacious. It is merely meant to illustrate one of the cognitive difficulties arising out of mathematical clarity.

Secondly, and more importantly, matter meant may be lost because of the very attribute that makes Tonnessen's definition so mathematically powerful, namely, its scope. Tonnessen pares away the "irrelevant attributes" of symbols which are those attributes that do not dictate whether a symbol is a variable or not and bases his definition solely upon that which is common to all variables. However, it is just those irrelevant attributes that distinguish different types of letter usage. The irrelevant attributes are completely relevant to the understanding of what is meant by variables.

Tonnessen would like to understand <u>variable</u> in the abstract. He has tried to pare away those effects of the contextual environment that influence one's cognitive understanding and contaminates the purity of the abstraction. According to Kaput (1979), however:

> The abstract mathematical formalism...gets its meaning via an anthropomorphism. We project or superimpose on the mathematical formalism, our own cognitive experience... When we use formal mathematics, we actually use the mathematics enriched with the context from cognitive experience. (p.292)

Thus to understand the mathematical formalism, variable, it is essential to understand those very relevant aspects of the context that are reflected in one's cognitive experience.

Tonnessen, in describing what he means by an irrelevant attribute, uses the analogy of the concept of trees. He says, "'is deciduous' is an irrelevant attribute for tree since an example of tree may or may not be deciduous." (p.13) Tonnessen thus, to coin a cliche, is, in order to define the forest, diminishing the importance of the identifying attributes of the trees. However, one cannot understand "matter meant" as it pertains to the concept forest unless one understands much about the distinguishing features of trees.

Tonnessen lists five irrelevant attributes. (pp.13-23) These attributes, though possibly irrelevant to Tonnessen's generalized definition, make up much of the "matter" that must be understood with regard to the use of letters in equations. They are as follows:

1. Kind of domain. Whether the domain of a symbol is a set of numbers or a set of things is irrelevant in determining whether that symbol is a variable.
2. Kind of variable expression. Whether the symbol is found in an open equation, a functional representation, or isolated phrase is irrelevant in determining whether that symbol is a variable.

3. Whether a symbol, such as "a," is labeled "constant," in an expression such as $ax + by + c = 0$, is irrelevant in determining whether "a" is a variable.
4. Whether an element of the domain of a symbol is a solution for an equation is irrelevant in determining whether the symbol is a variable.
5. Whether an element in the domain of a symbol when substituted results in an undefined expression is irrelevant in determining whether the symbol is a variable.

To illustrate how critical these irrelevant attributes can be in determing "matter meant" consider the following three equations. In each case, the goal is to say what is meant by the letter C. In each case, assume that the domain of C is I, the set of positive integers.

Equation 1. $C = 3 \times 2$

Equation 2. $C = 3 \times L$

Equation 3. $C = 3 \times L$

According to Tonnessen's definition, the variable (C,I) is the same for each of the three equations. What distinguishes C in equation 1 from the C in the other two are irrelevant attributes. These attributes are number 2 (they are in different kinds of expressions) and number 4 (there is a numerical solution for C in equation 1). Since equations 2 and 3 are identical, and since the C's in both cases have the same domain, they obviously cannot be distinguished from each other.

Suppose, however, that equation 2 gives the relationship between C, the number of bags of cement, to L, the number of bags of lime in a particular mortar mix. Suppose equation 3, on the other hand, gives the relationship between bags of cement and bags of lime in any mortar mix, in general. In both cases, one imagines that C can take on any element in its domain. That is, one allows C to vary over the positive integers. Furthermore, the solution set for each equation is, in the abstract, the following set of ordered pairs.

$$S = [(1,3), (2,6), (3,9), ...(n,3n)....]$$

Yet there is clearly a verbal difference between the two situations that these equations model. There seems also to be a cognitive difference in the two problem situations. For equation 2, the "empirical solution set" is just one of the ordered pairs contained in set S above. The problem deals with a unique mortar mix which must therefore have a fixed number of bags of lime and cement. The empirical solution set is the unique ordered pair that matches that situation. For example 3, the "empirical solution set" is the same as set S above (with the possible inclusion of an outer bound). Equation 3 can be thought of as a recipe that could be used for any mortar mix, thus any of the ordered pairs apply.

The difference then in what C means in each case is that in equation 2, it stands empirically for a unique value whereas in equation 3, C varies over the positive integers. This difference cannot be accounted for by any of Tonnessen's irrelevant attributes. One might argue that irrelevant attribute 3 applies; that in equation 2, C is really being thought of as an arbitrary constant. However, what Tonnessen refers to as arbitrary constants are those constants used in conjuction with other variables (as in $ax + by + c = 0$) where "one assumes that a, b and c are intended to be evaluated prior to the variable x [and y]." (p.16)

That the distinction between the usage of the symbol C in equations 2 and 3 was not anticipated by any of the five irrelevant attributes is not necessarily an indictment of Tonnessen's work. The purpose of including those examples at this point is to underscore the importance of going beyond a definition to understand the uses of letters in equations. To ascertain "matter meant," one must move from the general and analyze the specifics. The approach, described below, that was taken by Küchemann is an attempt to do just that.

Küchemann's Specific Categorical Approach

Dietmar Küchemann, in "Children's Understanding of Numerical Variables" (1978), tries to assess "matter learned," that is, students' conceptions concerning the use of letters in algebra. Like Tonnessen, he first develops a bench mark, matter meant, against which he can measure students' conceptions. Unlike Tonnessen, his bench mark is not based in a generalized definition of variable but is a somewhat hierarchical taxonomy of the different uses of letters in algebra.

His work has been strongly influenced by the work of Collis (1975) in at least two ways. First of all, both men use the Piagetian model for cognitive development in that they associate certain student conceptions with different Piagetian levels. Secondly, Küchemann's six leveled taxonomy is based on a less refined and detailed taxonomy given by Collis. Collis' taxonomy is not meant to be a description of matter meant but solely an analysis of conceptions of students at different concrete operational and formal operational levels. He has described three views of "pronumerals" (numerically valued letters) that correspond with three different concrete operational levels. A child at the first level has the view that pronumerals "map directly into (a single) specific number" (p.iv) and cannot be replaced, even experimentally with more than one number. At the next level, the child is able to plug in and test a series of numbers for the pronumeral but s/he does so with the idea that there is still a unique correct replacement. The final concrete operational level described by Collis is what he refers to as the level of concrete generalizations. At this stage the student has developed the concept that a letter is a "generalized number." This student would be able to use the formula for a rectangle, A = L x B, and would understand that this formula would work for any rectangle. But, as Collis says,

> for him, [it] is essentially a single system of co-variation; e.g.; the area changes as the rectangle changes. What he cannot do is relate changes in one or more of the variables "A" and "L" and "B" to changes in one (or more) of the others, e.g., he would not be able to solve problems of the kind, "A" is to stay constant and "B" is to be changed in some way, what must be done to length "L"? (p.9)

To be able to perform the latter task requires a level of abstraction with which only "formal" thinkers are capable of dealing. By way of further explanation Collis says that those acting at the top concrete operational level:

> seem to have extracted a concept of "generalized" number by which a symbol "L" (say) could be regarded as an entity in its own right but having the same properties as any number with which they had previous experience. At this level the symbol seemed to operate as an abstraction not yet in the form of a variable as such because it was still dependent on both the domain of numbers in the range where they had had experience and the kinds of operations which belonged to their number experience. (p.44)

It is essentially those four levels of conceptions pertaining to letters in algebra that have been modified and expanded into Küchemann's six levels. Küchemann differs with Collis in at least one significant way in that he had attempted not only to categorize student conceptions but also specific problem types. That is, he has developed a taxonomy that classifies problems as well as students. According to Küchemann (1978, p.23), the six classes are as follows: The examples given are those he uses to illustrate each level.

1. Letter Evaluated.

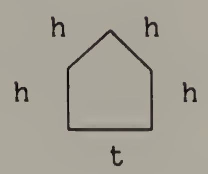
   Example 1. If a + 5 = 8, then a = ? "Here 'a' can be evaluated immediately; there are no intermediate steps involving an unknown."

2. Letter Ignored.

   Example 2. If a + b = 43, then a + b + 2 = ? "The second equation differs from the first by the term + 2; a + b can be ignored."

3. Letter as Object.

   Example 3. What is the perimeter of the following figure. "h and t are names or labels for the sides rather than numbers."



4. Letter as Specific Unknown.

   Example 4. The perimeter of an n sided figure where the length of each side is 2. Here, "n stands for an unknown number which cannot be evaluated."

5. Letter as Generalized Number.

   Example 5. If c + d = 10 and c<d then c = ?

   "c represents a set of numbers rather than just one value."

6. Letter as Variable.

Example 6. "What is larger, 2n or n + 2?" "A second order relationship needs to be found between 2n and n + 2 as n varies."

By way of further illustration, recall the "mortar mix" problems on page 24. In the second equation, C = 3 x L, which gave the relationship of cement to lime in a particular mortar mix, C would be thought of as a "specific unknown." When C = 3 x L is regarded as a recipe for all mortar mixes, C is a "generalized number." Finally, according to Küchemann, C would be regarded as a "variable" if one asked, for example, what would happen to C if L were doubled.

Both Küchemann and Collis have made significant contributions towards an understanding of the way letters are used in algebra. They imply that it is not sufficient to lump all letters under the umbrella term, "variable," without recognizing the qualitative distance that exists between letters found in different algebraic contexts. However, there are some difficulties with, and limitations to Küchemann's taxonomy that will be addressed in the following:

1. Küchemann, in places, does not make an adequate distinction between "matter meant" and "matter learned." The tone of his taxonomy is one of matter meant, that the "level" of letter use is inherent in the structure of the

problem. For example, in example 2, it is not improper or ill-conceived to "ignore" the a + b. Nor is it wrong to regard the n in example 4 as being a "specific unknown." The structures of those problems allow for, if not demand, those conceptions about the letters. However, in example 3, it is wrong to regard the h and t as being "objects." Küchemann has indicated that when students do so, they give answers like 4ht and hhhht instead of the appropriate 4h + t. The latter correct answer is predicated on an understanding of h and t as numbers representing the lengths of the corresponding sides. (Küchemann, in a later paper, "Object Lessons in Algebra?" (1981), is more explicit about the problems that arise from viewing letters as objects.)

Students' tendency to regard letters as objects is a misconception that will be discussed in later chapters. It is certainly a prevalent misconception and the data that Küchemann has collected pertaining to it are revealing. However, as mentioned above, it is important to develop a mechanism for understanding letter use independent of students' conceptions in order to have something with which to compare those students' conceptions.

There are some situations where one can legitimately use letters to stand for objects. This strategy can be found in the process of doing unit analysis in physics and elsewhere. But this usage was not implied by Küchemann's description and examples of this level.

2. Küchemann overemphasizes the hierarchical nature of his taxonomy. Though his data have clearly shown that problems involving, say, letter as "generalized number" or letter as "variable" are more difficult for students than problems involving, for example, letter as "specific unknown"; and though it is probably true that the former types of problems require more formalized thought than the latter type, one should not infer from that, that to always think of letters in terms of Küchemann's idea of "variable" is <u>necessarily</u> more sophisticated and preferable to thinking of the letter as, for example, "specific unknown." Küchemann does acknowledge that

> a problem [that] is often meaningful only if the letters are understood at a high level of interpretation...[is] often, and quite legitimately, solved by switching to a lower level (from variable to specific unknown, or specific unknown to object) during intermediate steps of manipulation for the very good reason that this reduces the amount of information, the cognitive load, that has to be carried. (p.26)

But it should also be emphasized that there are some problems, like examples 2 and 4 described above, that should be approached from the start with a lighter "cognitive load." There is often no reason to view the letters at a higher level of abstraction than is called for in the empirical context of the problem.

3. Küchemann does not pay enough attention to the causal effects that the context of a problem has on the

level at which a letter is classified. He does acknowledge that

> consideration must be given to the context in which the levels are being used: the child's likelihood of solving any given item will depend not only on the level of interpretation that the item requires but on the interaction of this with other dimensions such as the type of operation involved (Brown and Küchemann,1976), the number of operations (Collis, 1974) and so on -- in other words, on the complexity of the item. (p.24)

One can infer from the above quote that the "level of interpretation" of a letter is in some way independent of the context in which it is found; that is, that the levels of interpretation can be defined and distinguished independently from the context of a problem.

However, the six defining examples Küchemann uses to illustrate his taxonomy are all quite different in terms of the problem context. Some examples demand a numerical solution; some do not. Some demand an equation as a final answer; some do not, and so on.

But are levels of interpretation independent of the problem context? It could be argued, for example, that in example 1, (if a + 5 = 8, then a = ?) and example 4, (find the perimeter of an n sided figure where the length of each side is 2), the different level of difficulty is attributable to the difference in the problems' context.

Küchemann himself would argue that problem 4 is more difficult than problem 1 partially because it demands Collis' "acceptance of lack of closure." As mentioned earlier, this is a phrase that Collis (1978) has coined to refer to the ability to feel content with an open ended answer like 2n (for example 4). Collis claims acceptance of lack of closure is an indicator of formal thought, and believes that if a student has not reached the appropriate degree of conceptual development, s/he would find problems requiring the acceptance of lack of closure very difficult.

Furthermore, example 4 may be more difficult than example 1 simply because it is a word problem that requires translations between symbol systems. In Küchemann's own words, it has more "cognitive load" to begin with.

It is unclear, then, how letter usage is different in these two examples. Therefore, since they are defining examples of supposedly different levels of letter use, it is unclear whether such distinctions are legitimate.

Tonnessen has tried to understand the concept variable independently from what he calls "irrelevant attributes," which pertain to the context of the problem. Küchemann's understanding of letter on the other hand seems to be highly dependent on those "irrelevant" attributes. He changes the domain, the type of expression, the type of solution, etc. in order to illustrate differences in letter use. Yet he seems to downplay the extent to which the domain, the type of expression and the type of solution, etc. do determine how a letter should be understood.

4. Küchemann says, with regard to letter as "variable" that "interpreting letters as variables involves an awareness that there is some kind of relationship between the letters, as their value changes in a systematic manner." (p.26) Though Küchemann refers to a "relationship between the letters," he is not meaning to imply that one must have at least two letters in an expression to have "variable." Example 6 (which is larger, 2n or n + 2?) which, according to Küchemann requires an understanding of letter as variable, has only one letter. But as the value of n "changes in a systematic manner" one must be aware of the "relationship between" the expressions, 2n and n + 2.

It seems then that a relationship of some sort is a requirement for a situation to be considered to contain "variable." Again, the definition of the class is highly dependent on the type of context in which it is found. It can be argued, however, that a "variable" is a symbol whose "value changes in a systematic manner" and that it exists independently from any relationship to other letters or expressions. If a + 5 = 8 were thought of as an open expression, one could systematically change the value for "a" in search of a value that is a solution to the equation. That is, "a" can be thought of as a variable. It seems that the hypothesis, conceiving of "a" as something that systematically changes is more difficult than thinking of it as being a single number, has merit.

However, the data that Küchemann uses to substantiate that are complicated by the fact that example 6 employs, in the words of Collis (1978), "multiple interacting systems." Collis believes that a problem that requires an "interaction between two systems" (p.227) has a prerequisite of formal operational thinking. Thus, "variable," as defined by Küchemann, might be a more sophisticated concept only because it is found in multiple interacting systems.

5. No distinction is made between letters that vary discretely and letters that vary continuously. Küchemann writes of "systematically" changing the value of a letter. However, there are at least two distinctly different "systems" that can be used. One can substitute values for the letter one at a time (i.e., discretely) or one can imagine the value of the letter changing smoothly and continuously. This issue will be raised again later in the dissertation.

6. Küchemann implies that a student's level of understanding of the use of letters can be determined by assessing which level are the levels of the hierarchy with which they are capable of dealing. One can infer from that that this taxonomy is a comprehensive chart of students' conceptions. This is not the case. Some of the conceptions and behavior patterns that will be discussed in later chapters are neither predicted or explained by

Küchemann's taxonomy.

A Mechanism to Assess Matter Meant

Tonnessen has tried to understand letter use independently from context dependent factors ("irrelevant attributes"). Küchemann, on the other hand, has recognized that letters take on very different meanings when couched in different problem contexts. However, he has not discussed what, in the context of the problem, causes the letters to take on those different meanings. He has created a taxonomy of letter use that seems highly context dependent without focusing on what those crucial factors are that influence the various categorizations.

Ideally, what is necessary is not a taxonomy of letter use, but a taxonomy of problem type vis-a-vis how they use letters. Such a taxonomy, however, because of the myriad of types of algebra problems, would necessarily either have broad and overly generalized classes or else would be broken down into literally thousands of different types. However, though it may not be practical to classify problem types, one can still develop a mechanism by which one can analyze problems and their use of letters. The following is a list of five questions that serve as such a mechanism. An hypothesis (that could provide the basis of further investigation) is that almost any difference between two algebra problems vis-a-vis their use of letters can be explained by variations in the answers to one or more of these five questions. Note that some of these questions parallel some of Tonnessen's "irrelevant attributes."

1. What is the abstract domain of the particular letter in the problem.
2. What is the dimension of the problem vis-a-vis a particular letter? Put differently, how many letters in the problem (including itself) is a particular letter actively related to?
3. How is a particular letter perceived to vary?
4. What is the solution set for a particular letter?
5. Is there some semantic meaning attached to a particular letter that is different from its meaning in its abstract form?

The following are descriptions of each of these five questions.

1. What is the abstract domain of a letter? The domain of a letter is the set of numbers (or things) which comprise all of the possible values that the letter can take on. In high school algebra, it is almost always a set of numbers and usually the set of real numbers.

The purpose of the word "abstract" is to exclude from consideration at this stage semantic characteristics of the referents of the letter. Thus if x stands for the number of apples and y stands for the number of people, the abstract domain is the same in each case: the non-negative integers.

It is important to understand differences in the domain that are somewhat independent from (or masked by) the semantic context of a problem. Whether the domain is made up of a continuous set (the real numbers) or a discrete one (the integers); whether, indeed, the domain is numerical or not are all important considerations in analyzing the meaning of a letter.

2. What is the dimension of the problem? Consider the following problem. "Find the length of a rectangle where the length is twice as long as the width and half of the perimeter is 21 feet long." The following solutions to this problem differ in terms of the number of letters that are actively (and consciously) interrelated.

Solution no. 1.

Let l = the length and let w = the width

Then $l = 2w$ and $l + w = 21$

Thus $2w + w = 21$

And $w = 7$ and $l = 14$

Solution no. 2.

Let x = the width Then $2x$ = the length

$2x + x = 21$

Thus $x = 7$ And the length = 14

The first solution employs explicit functional relationships (e.g., $l = 2w$) where one letter is actively compared with another. The second solution avoids functional relationships and relates a single letter to concrete entities. The first involves the solution of a system of simultaneous equations, the second does not. Viewed in the abstract, an equation like $l + w = 21$ represents a line whose solution set is an infinite set of ordered pairs. On the other hand, viewed in the abstract, the equation $2x + x = 21$ has a solution of a single number.

Increasing the dimensionality often represents, as Collis might say, an increase in the covariance of letters and expressions. A functional relationship is the covariance of the dependent variable with the independent variable or variables.

Determining the dimensionality of a problem can be a subjective task. As shown above, problems can be done in different dimensions. Furthermore, a single equation can be viewed in different ways. Consider the equation $y = mx + b$. If seen as a general equation for a line, the equation is two dimensional. One is not actively interrelating the m and the b with the y and the x. One fixes them as constants and then thinks of the covariance of the y and the x alone. On the other hand, $y = mx + b$ can be thought of as a four dimensional problem, a function of y in terms of the three independent variables m, x and b.

As mentioned previously, the meaning of a letter is largely determined by the context within which it is found. If that contextual environment includes a relationship with another letter or letters, that significantly influences its meaning. Thus dimensionality plays an important part in understanding what is meant by a letter.

3. How is the letter perceived to vary? A letter can be thought of as taking on different values in its domain. That is, it can be perceived as varying over its domain. However, depending on the context of the problem, the letters can vary in different ways.

Consider again the formula for a line, $y = mx + b$. Assume that the domain of all of the letters is the set of real numbers. In imagining the drawing of the graph of such a line, one can imagine sweeping <u>continuously</u> over all of the points (x,y) that make the equation true. One imagines that x is replaced continuously with all the values in its domain. However, when one fixes m as constant, m does not vary, cognitively. In thinking of a particular line, m stays fixed. Though m has a domain of all the real numbers, m is thought to (temporarily) take on only one of those values.

In many problem contexts, letters are perceived as varying discretely. This can occur when the domain itself is discrete (e.g., the integers). However, it can also occur when the domain is a continuous set. Time, which is a continuous entity, is often thought of discretely (in terms of the number of seconds or hours). In some problem contexts (where continuity is not an integral facet of the problem) it is often sufficient, if not more efficient, to think of letters with continuous domains as varying discretely.

Thus the meaning of a letter is in part determined by whether it is perceived not to vary, to vary discretely, or to vary continuously.

4. What is the solution set for the letter? The answer to this question depends in part on the dimension of the problem. In the case of problems in one letter, the following examples illustrate widely differing solution sets:

	Solution Set	Characteristic of Set
a. $x - 1 = 0$	[1]	One element
b. $x^2 - 1 = 0$	[1, -1]	Finite no. of elements.
c. $x - 1 < 0$	[x/x < 1]	Infinite no. of elements.
d. $x + 0 = x$	All real numbers	Entire domain.
e. $x^2 = -1$	[$\emptyset$]	The empty set
f. $x - 1$	Solutions are irrelevant to this expression	

Note that though all could have the same domain, their solution sets are quite different. Note also that for example f., it is irrelevant to talk of solution sets. Many problems demand as a final answer an equation or expression. Numerical solution sets are often not relevant to the context of the problem.

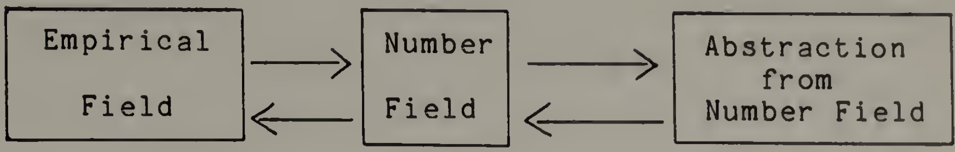
Problems in two letters are also subject to categorization according to solution set analogous to a - f above.

	Solution Set	Characteristic of Set
a. $x - y = 1$ $x + y = 1$	[1, 0]	One ordered pair
b. $x^2 - 1 = 0$ $x + y = 1$	[(1, 0), (-1, 2)]	Finite no. of ordered pairs
c. $y = 3x + 4$		Infinite no. of ordered pairs
d. $x(y + 1) = xy + x$	[R X R]	All ordered pairs of real numbers
e. $x^2 + y^2 = -1$	[$\emptyset$]	The empty set
f. $x + 3y$	Solutions are irrelevant to this expression	

Two letters in two separate problems for which identical answers to the first three questions are given would still have different meanings if their solution sets differed in any of the ways illustrated above. Thus the solution set (or lack thereof) is an important consideration in understanding letters.

5. Is there some semantic meaning that is different from the abstract meaning of the letter? Mathematics in general, and algebra in particular are abstract models or tools which can be employed to better deal with problems that are couched in a more concrete context. Collis (1974,

p.11) has described a path which the solution of a problem will often follow and uses the following illustration.



An illustrative example is that of finding a confidence interval for a mean in Statistics. One would like to estimate the mean of some attribute for a population. To do so, one takes a sample from the population (the empirical field) and finds the mean and standard deviation of that sample (the number field). One would like to understand something about the mean of the population based on the concrete results of the mean and standard deviation of the sample. The way this is usually done is to analyze the behavior of means of samples in general (i.e., in the abstract field). If one were (hypothetically) able to find an infinite number of means of samples, one would find that they would be distributed normally (the bell shaped curve). Submitting the concrete results (the mean and standard deviation of the sample) to abstract analysis (the normal distribution) allows one to determine a probable range of values in which the mean of the entire population will fall.

This process of submitting concrete data to abstract analysis is an integral part of the application of algebra to real world situations. When one is given an algebra word problem, (empirical and/or number field) one often symbolizes the information in algebraic expressions (the abstract field). When that is done, the letters used have semantic meaning attached to them as well as abstract meaning. I will refer to such letters as semantically laden letters. Semantically laden letters stand for what Schwartz (1976) has called "adjectival numbers." Adjectival numbers refer to those numbers that are used as adjectives such as four books or ten women and are contrasted, according to Schwartz, with "nominal" numbers which are those numbers that are used as nouns. Thus if x stands for the number of books, it takes the place of an adjectival number and is thus semantically laden.

(At the Dimensional Analysis Project of the Education Development Center in Newton, Massachusetts, Schwartz (1981), and his colleagues are developing a curriculum that attempts to get students to clearly distinguish between the quantitative and semantic referents of a letter or number. Students are required to ask two separate questions, "how many" and "what," to make that distinction.)

Consider, again, the examples pertaining to mortar mix on page 24. When the equation $C = 3L$ is taken in isolation the letter C takes on its abstract meaning--an element of the set of numbers comprising its domain. When it is seen as in the second equation of page 24, as the relationship between the number of bags of cement and the number of bags of lime in a mortar mix, the letter C becomes associated not only with its abstract domain but also with its semantic referent, cement. It becomes a semantically laden letter, and thus, on some level, must be viewed differently from a letter with only abstract meaning. Student conceptions of semantically laden letters is the focus of the empirical research presented in Chapters IV and V of this dissertation.

One aspect of semantically laden letter that will not be discussed in those chapters but is nevertheless pertinent to a discussion of "matter meant" is the "distance" between the semantic and abstract meaning of a letter from one problem context to another. As pointed out on page 24, when C = 3L represents the relationship between the amounts of ingredients in a particular mortar mix rather than a recipe for mortar mixes in general, the semantic meaning of C in each case changes. It changes from a conception of a single but unknown number (Küchemann's specific unknown) to that of representing any element from a set of numbers (Küchemann's generalized number). Because, <u>in the abstract</u>, C represents any element from a set of numbers, the semantic meaning of C, when the equation represents a recipe, is "closer" to the abstract referent than when C refers to the number of bags of cement in a particular mortar mix.

No attempt will be made to define a metric that determines "distance" between the semantic and abstract meanings of a letter nor is any claim being made that such a metric is possible. Nevertheless, the degree to which an abstract symbol is removed from its concrete referent must play a significant part in determining its meaning.

One other complication that should be noted in trying to assess the meaning of a letter within the context of a problem is that at different stages in the problem, the meaning of the letter vis-a-vis the above five questions, can shift. At one stage of the problem, the expressions may be two dimensional whereas at another stage, they can be viewed as one dimensional. At one stage of the problem, the letters can be viewed as varying continuously whereas in another stage of the problem, it might be necessary to hold the letter constant.

Another consideration is that, in some problems, some of the questions may not be answered definitively. In some problems, it is irrelevant whether the letter is varying discretely or continuously and either view is acceptable. In some problems ($y = mx + b$ was an example) the dimensionality of the problem is somewhat dependent on the view of the problem solver.

Despite the lack of hard and fast categorizations, these five questions remain a powerful tool in comparing problem types vis-a-vis their use of letters. To illustrate, Küchemann's six examples will be analyzed using the five questions.

1. Problem: $a + 5 = 8$
   $a = ?$

   Domain? Presumably the real numbers.

   Dimension? One.

   Variation? Unvarying.

   Solution Set? One element.

   Semantically laden? No.

2. Problem: $a + b = 43$
   $a + b + 2 = ?$

   Domain? Presumably the real numbers.

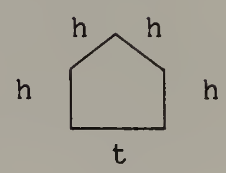
   Dimension Two or none.

   Variation Unvarying.

   Solution Set? Irrelevant.

   Semantically Laden? No.

3. Problem: What is the perimeter?



   Domain? The real numbers.

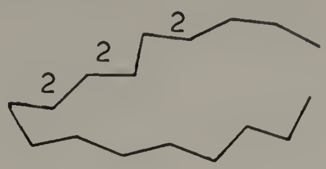
   Dimension? One-- because the letters do not covary with each other

   Variation? Discrete.

   Solution Set? Irrelevant.

   Semantically Laden? Yes.

4. Problem: What is the perimeter of an n sided figure, all sides the length of 2?



Domain?	Positive integers.
Dimension?	One.
Variation?	Unvarying.
Solution Set?	Irrelevant.
Semantically Laden?	Yes.

5. Problem: $c + d = 10$
$c < d$
$c = ?$

Domain?	The real numbers.
Dimension?	Two.
Variation?	Continuous (c) and unvarying (d).
Solution Set?	Infinite number of solutions.
Semantically Laden?	No.

6. Problem: Which is larger, 2n or n+2?

Domain?	Real numbers or integers (n often stands for integers).
Dimension?	One.
Variation?	Discrete (but open to other interpretations.)
Solution Set?	Irrelevant.
Semantically Laden?	No.

Küchemann has made a big step in recognizing that letters can be interpreted differently in different problem contexts. What he has not done is to be clear about what factors of that context influence the different interpretations. The analysis of a letter, via the five questions, helps to delineate those factors. As can be seen above, the differences of the problems, vis-a-vis their use of letters, is explained by the differing answers to those questions.

The purpose of this chapter has been to try to better understand what is "meant" by letters in algebra. The initial answer to that question is that it depends. It depends on the problem context. The factors within that context that lend meaning to the letters are the subjects of the five questions: domain, dimension, variability, solution set, and semantic meaning.

How those five subjects are dealt with in the text books will be the focus of the next chapter on "matter taught."

CHAPTER III

MATTER TAUGHT: A TEXTBOOK REVIEW OF THE PRESENTATION AND USE OF LETTERS IN ALGEBRA

In the previous chapter, five questions were presented and discussed that underscored five basic attributes of letters found in algebra problems. These five questions comprise an analytical device with which one can help determine how a letter is being used and what it means in the context of a problem. That is, they help one to understand "matter meant."

As Bauersfeld pointed out, however, "matter taught" is certainly not always the same as "matter meant." Even if the author of a text or a teacher are themselves aware of the crucial issues and fundamental concepts of a topic (which is not always the case), they may not be capable of teaching these issues and concepts in such a way that the student gains some kind of understanding. Often, the subject gets presented in such a way that students come away from the experience with confusing misconceptions or misleading simplifications.

This chapter is a review of how forty-one high school, junior high school, and college algebra and mathematics textbooks present the use of letters in algebra. The bibliography of the publications reviewed (see Appendix C) contains recent editions up to and including some published in 1980 from several of the large and popular publishing houses as well as some texts used in the 1960's and early 1970's. In addition, several experimental and/or innovative texts were reviewed including some from the Stanford University School Mathematics Study Group (SMSG), the University of Illinois Committee on School Mathematics, the Madison Project, and a 1980 innovative text by Robert Foerster. The list includes all those relevant publications obtained as a result of an Eric search conducted in 1980 that was made in search for articles and texts cocerning the concept of a variable. Though the bibliography is not exhaustive, it presents a wide spectrum of styles of presentation and content. This review thus becomes a window to various means of teaching letter usage in algebra: i.e., matter taught.

How various texts deal with (or do not deal with) the subjects of the five questions described in the previous chapter will be discussed in the first five sections of this chapter. A sixth section will be included which will review to what extent the texts' presentation of letters in algebra is confusing and/or ambiguous.

Almost all texts reviewed refer to letters used in algebra as variables. As will be seen, not all texts imply that variables vary in some way. Nevertheless, to be consistent with the texts being reviewed, in this chapter, the word variable refers in general to letters used in algebra.

The Domain or Replacement Set

What constitutes the replacement set (or domain) of a variable? A replacement set is made up of all those elements that can be substituted for a variable. In High School Algebra, the replacement set for a variable is almost always made up of numbers, usually the entire real number line. However, in Statistics and elsewhere, one often talks about qualitative, as opposed to quantitative, variables, where the replacement set can be made up of words. In differential equations, the variables represent functions and in Topology, the replacement set for the variables is often sets of sets, or sets of sets of sets. Does one then, when introducing the concept of variable, present a general definition that encompasses all possibilities for replacement sets, or does one simply say that variables will stand for numbers, thereby making the concept less abstract? This is one way in which the textbooks differ. The following are examples of three common ways of describing replacement sets. (The numbers below refer to columns in Tables 6 and 7 of Appendix D.)

1. The replacement set is a set of things. One of the most generalized definitions of a variable is found in High School Mathematics by Beberman (1960). "A variable is a pronoun" and can be replaced by the name of "numbers,

sets, points, people, teams, etc." (p.9) (Beberman does make a distinction between variables that are replaced by numbers and refers to such variables as pronumerals.)

2. *The replacement set is a set of numbers.* An example is given by Traves (1977) in *Using Algebra* who define a variable as representing "any number in a replacement set" of numbers. (p.78) These two definitions are further contrasted by:

3. *Variables are replaced by one number.* Nichols, et al (1978) in *Holt Algebra I* simply say that a variable "n" "takes the place of a number." (p.4) There is no reference to sets and there is no sense of the multi-valued nature of a variable. Tables 6 and 7 (found in Appendix D) catalogue the texts in terms of how, in the definition of variable, replacement sets are described. Included in these tables are categories 4 and 5.

Category 4 includes those texts with unorthodox approaches that cannot be categorized into 1, 2 or 3 above. For example Davis (1964), in work done at the Madison Project, uses symbols like $\square$ which he calls "a placeholder" for a number or thing. (p.25) Krause (1964) in *Mathematics I: Concepts, Skills and Applications* defines a variable only in the context of probability. (p.32) Haber-Schaim, Skvarcius, and Hatch (1980), in *Prentice-Hall Mathematics*, use a "window" approach similar to Davis' above while also introducing variables via arguments involving combinatorics. (p.243)

Category 5 is made up of those texts which gave no explicit definition of variables.

Dimension

In what dimension are letters presented? As mentioned in Chapter II, many of the word problems that are presented in algebra texts can be done both with one or two variables. Rich (1973), in <u>Modern Elementary Algebra</u> (a Schaum's outline), demonstrates both methods side by side. (p.190) An abridged example is as follows:

> The larger of two numbers is three times the smaller. Their sum is eight more than twice the smaller. Find the numbers.

Let s = smaller number $3s$ = larger number	Let s = smaller number l = larger number
Then $3s + s = 2s + 8$ So $s = 4$ $3s = 12$	Then $l = 3s$ $l + s = 2s + 8$ So $s = 4$ $l = 12$

Most texts opt strongly for the one variable solution. In these texts, all problems that can be, are done with one variable, even when a two variable approach would be less awkward. In so doing, the explicit functional relationship between the two variables is circumvented. It is this functional relationship that brings out the dynamic quality of variables. That is, in the language introduced in the previous chapter, to understand the functional relationship between two variables, one must be able to perceive the discrete variability (if not the continuous variability) of the respective variables.

To further explain how a one variable solution to a problem is significantly different from the two variable solution, consider the following version of the Students and Professors problem (which was first described in Chapter I.)

> There are six times as many students as there are professors. If there were 10 professors, how many students are there?
>
> (The answer is 6x10 or 60 students.)

The solution to this problem is analogous to that of a one variable solution of the original problem, which might look as follows.

Let P = number of professors.
If there are P professors, then there would be 6P students.
So 6P = number of students.

Note that, though "6P = number of students" is equivalent to 6P = S (since S is the number of students), the former representation avoids the explicit comparison of two discretely varying variables.

Among the same people who did poorly on the original Students and Professors problem, over 95% were able to solve the arithmetic version, according to Clement (1981). It appears that the "path of least resistance" would be to opt for the one-variable approach to word problems because by doing so, one may be cognitively "closer" to the more simple arithmetic approach. However, by avoiding functional relationships, one avoids one of the fundamental concepts of mathematics.

Very few texts made a connection, as Rich did, between one and two variable solutions to the same problem. If that connection is not made, it is possible that those word problems that come later in the curriculum that do require the use of functional relationships may appear foreign and unreasonably difficult to the students.

Tables 8 and 9 of Appendix D were filled out conservatively for this category (column number 5) in that only those texts that virtually excluded any two-variable representations in their solutions to word problems were included.

Variability

What impression of the manner in which variables vary does the book convey? There are at least three different possibilities. (The numbers below refer to columns in Tables 6 and 7 of Appendix D.)

6. Nonvarying variables. This is what is often referred to as an unknown. As mentioned in the previous chapter, it can also be referred to as a constant in the equation $y = mx + b$. It is a way of conceptualizing a variable that is similar to when one solves a word problem for which there is a unique answer or, at most, finitely many answers. Consider the following problem:

Find the length of the side of a square whose perimeter is 20 inches.

One might solve this problem as follows: "Let x be the length of one side. Then 4x = 20. So x = 5." At the stage where one thinks "x is the length of one side" and despite the fact that the variable x has an infinite replacement set (all of the positive real numbers) one conceptualizes a unique square and the side of that square does not change (Küchemann's specific unknown).

Many textbooks only present word problems that have a single numerical answer. In these, the student may infer that the only appropriate replacement for a variable is a single number. Either variables are not replaced when found in expressions like "x + 5" (where there is no solution) or the x is only replaced once. Dilley and Rucker (1975), in <u>Heath Mathematics</u> gives an example like the following. (p.4) The expression (A + a) - n is given along with the following table:

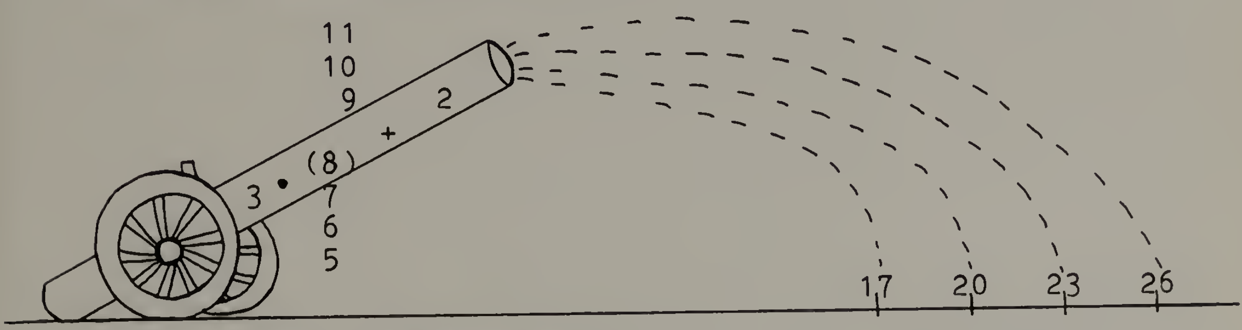
A	a	n
5	2	4

The replacements are then shown: vis. (5 + 2) - 4. Again, the implication is that each variable has exactly one appropriate replacement.

7. <u>Discretely varying variables</u>. Considering variables in functional relationships differs from the "static" situation described in f above. The equation, 12x = y, relates the number of eggs in a store to the number of

full egg cartons in that store (a dozen eggs per carton), where x is the number of cartons and y is the number of eggs. The expression calls for one to conceptualize infinite (or at least a large number) of possible extensions of x.

One way of demonstrating the discrete nature of the variability of a variable is with tables, a technique used by several texts. Another example is the following. Pearson and Allen (1964, p.60), in Modern Algebra --A Logical Approach, used a "cannon" to solve the expression 3x + 2 = 26. The idea is that as different values for x are fed into the cannon, the cannon shoots out a value for the expression, aiming for 26. (Note that it is not necessary to have an expression in two variables to demonstrate discrete variation.)



Again, the discrete nature of the replacements is emphasized.

8. Continuously varying variables. This is similar to the discrete situation except that one must conceptualize the variable changing continuously. Examples of continuously varying variables are those in the equation

$D = 40 \times t$ where D is distance and t is time. To understand that a variable can change continuously is an essential prerequisite to understanding the concept of the limit in calculus, and thereby calculus itself.

It can be confidently stated that no text fully explained the idea of a continuously varying variable. A few that came close did so by repeatedly using graphic representations for the functions of continuous quantities; or by emphasizing the use of variables in tautologies, stressing the fact that the letters could be replaced by _any_ number.

The difficulty in getting the concept of continuous variation across is compounded by the fact that the very act of replacement must be done discretely. That is, one would be hard pressed to demonstrate the continuous nature of the variable D in $D = 40 \times t$ by plugging in values for t. This dilemma is further exacerbated by the fact that though distance and time are conceptually continuous, any measurement of them must be discrete. What is continuous is the _image_ of a _moving_ point, and this is hard to symbolize on paper. Beberman (1960) comes close to dealing with this issue when he defines a "variable quantity." A variable quantity is a function whose range is quantitative. An example of a variable quantity is the function A. A is the set of all ordered pairs (x,y) where x is an element of the set of all squares and y is an element of the positive real numbers such that y is the area of x. (p.97) (Beberman calls both x and y variables, y being the type of variable called a pronumeral.) Because A is viewed as a function that changes as x changes, the continuous nature of y may still be lost.

It is an open question as to whether dealing with continuous entities discretely simplifies the problem or whether it is an oversimplification that loses the essence of continuity. In fact, it is possible that both approaches are too abstract for many High School students, thereby implying the need for an approach similar to F above.

Tables 6 and 7 of Appendix D under columns 6, 7 and 8 classify the texts according to how they tend to present the variability of variables.

Solution Sets

Is a solution set defined and is there a difference between the solution set of an expression and the replacement set of a variable? One potential difficulty with defining a variable as standing for any one of a given set of numbers becomes apparent when dealing with solvable word problems with one or two "unknowns." How, for example, can a variable stand for any one of the real numbers when there is only one or two correct solutions? At least 10 of the texts attempt to address this issue by defining a solution set that is different from the replacement set. They emphasized that distinction by noting that an open expression (e.g., $3x + 2 = 26$) is either true or false depending on which value of the replacement set one uses for x. Furthermore, any open expression is true for either some, all, or no values for x. Those values that make the expression true make up the solution set.

Semantically laden letters

How do text books present and use letters that have semantic meaning attached to them as well as abstract meaning? Semantically laden letters occur most frequently in word problems. In solving algebraic word problems, a student must translate more or less concrete data into more abstract algebraic symbolism. A letter resulting from such a translation has both semantic and abstract meaning associated with it.

What opportunities do text books give to students to develop skills in translation and experience in dealing with semantically laden letters? Is the texts' presentation of the translation process helpful in developing an understanding of such letters?

In reviewing the forty-one texts, a count was made of those problems which could be classified as word or "thought" problems. These are the problems for which letters used have more than abstract meaning. Of the forty-one texts, a majority (over 60%) devote less than 25% of their exercises to problems requiring more than the mechanical manipulation of expressions and equations. In fact, almost half of all of the texts reviewed devoted less than 10% of their exercises to these more involved "thought" problems. Furthermore, of the eleven Algebra I and Algebra II texts put out since 1975, only one devoted more than 20% of its exercises to word or thought problems. The one exception is a text by Foerster (1980), which devoted well over 50% of its exercises to word problems and problems involving critical analysis.

Consider, as an example, the problem of finding the equation for a line connecting two points. Most traditional texts approach this problem by providing a formula like the following:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

They then give a list of exercises that consists of pairs of points. The student solves each exercise by using the formula. Letters are devoid of semantic meaning. Foerster, on the other hand, after developing a strategy for finding linear equations from a pair of points, uses word problems as exercises. The student then not only practices the mechanical skill of finding linear equations but also continues to increase the analytical skills that go beyond the mechanical and is forced to deal with letters with semantic meaning. Foerster shows the student how linearity manifests itself in the real world. The following is an example:

> Cricket Problem
>
> Based on information in deep River Jim's Wilderness Trail Book, the rate at which crickets chirp varies linearly with the temperature. At 59 degrees, they make 76 chirps per minute, and at 65 degress they make 100 chirps per minute.
>
> a. Write the particular equation expressing chirping rate in terms of temperature.
>
> b. Predict the chirping rate for 90 and 100 degrees.
>
> c. Plot the graph of this function.
>
> d. Calculate the temperature-intercept. What significance does this number have in the real world?
>
> e. Based on your answer to part d, what would be a suitable domain for this linear function? Make your graph in part c agree with this domain.
>
> f. What is the real-world significance of the chirping rate intercept?
>
> g. Transform the equation of part a so that temperature is expressed in terms of chirping rate.
>
> h. What would you predict the temperature to be if you counted 120 chirps per minute? 30 chirps per minute?
>
> (p.56)

The contrast of this approach with that of, for example, Stockton's (1978) in Essential Precalculus is dramatic. In the algebra section of the latter book there are 1,000 problems, all of which require little more skills than memorizing formulas and mechanical techniques. There are only ten word problems, and those are only included as an appendix to the Algebra section with no instruction or introduction provided. (p.146) The student studying algebra via such a text gets virtually no experience with semantically laden letters. Those texts with fewer than 25% word or thought problems are indicated in Tables 8 and 9 under column 1 in Appendix D.

One student misconception (referred to earlier and to be investigated and discussed in later chapters) concerning letters with semantic meaning attached is that they overly associate the semantic meaning with the letter, at the expense of the abstract meaning. Thus, a letter B, which stands abstractly for some unknown <u>number</u> of books is thought also to stand for the word <u>books</u> or some other attributes of books. Three teaching strategies found in many texts might foster this over-association of the concrete referent of a letter onto its abstract meaning. These three strategies are described below:

1. <u>Word problems are first introduced with pure number problems.</u> An example of a pure number problem is the following:

> "Tom got an answer of twenty when he added four to two times his original number. What was the original number?"

In solving this problem, one might write "let x be the original number." Here x is, appropriately, directly associated with a number. Contrast this to the following problem:

> "Tom bought 20 books which is four more than two times the number of books he had originally. How many did he have originally?"

The concrete entity of interest in this problem is "books". However, x should not be directly associated with books but to some number (of books).

The hypothesis that an overemphasis on pure number problems fosters misconceptions concerning the role and meaning of variables in word problems is, at present, unsubstantiated. Nevertheless, it is an interesting characteristic for comparison. Texts whose first section on word problems primarily focuses on pure number problems were checked under column 2 in Tables 8 and 9 of Appendix D.

2. <u>The wording of problems is contrived, apparently to avoid mental effort</u>. Many textbooks seem to take the "path of least resistance" in terms of the wording and structure of word problems. In so doing, the texts often avoid situations where an over association of letter with concrete referent would cause confusion or disequilibration.

Consider the following two sentences.

The first number is twelve less than the second number.

The first number is equal to the second number decreased by twelve.

A correct equation for both sentences is $x = y - 12$ where x represents the first number. Notice that the latter sentence more easily maps into the equation. However, it is more awkward and contrived than the former. Many texts opted for the latter type of sentence over the former. They protect the student from a confrontation with the meaning of words, thereby reducing the mental effort required of the student.

As mentioned in Chapter I, Clement (1980) and Clement, Lochhead and Monk (1981) first documented what is referred to as the "reversal error." This reversal error manifests itself with, for example, the Students and Professors problem, which is recopied here for reference.

> Write an equation which represents the following statement. "At a certain university, there are six times as many students as there are professors." Let S be the number of students and P be the number of professors.

The correct answer is 6P = S. The reversed answer is 6S = P. (As mentioned in Chapter I, over 50% of all business and social science calculus students who tried this problem answered it incorrectly.) However, when the wording of the original problem is changed to "the number of students is six times the number of professors," the error rate drops dramatically.

Almost all of the texts avoided wording of problems in a way that would induce reversal errors, opting for the latter wording. This is true even though language like "there are six times as many students as professors" is common and comfortably colloquial. When it is found in the text, it is usually found in its additive version (there are six more of this than that.) Otherwise, it is found in the context of one-variable problems, which circumvents the reversal difficulty (as mentioned above).

Most of the text books reviewed (Foerster's being a notable exception) focused on word problems with short quantitative answers. Problems whose final answers are equations or verbal analyses were very rare.

By taking these paths of least resistance, the student is probably more successful (in terms of the number of correct answers) and therefore the teacher can (naively) feel more successful. But one must question whether, by reducing the need for mental effort, a text book is avoiding those situations where conceptual learning and development really do take place. The book may be avoiding those situations which force a student to truly understand how letters are used in algebra.

An x was placed under column 3 in Tables 8 and 9 of Appendix D if a large majority of the word problems (over 75%) were worded so as to fit into a pattern or to, in some other way, reduce the need for mental effort on the part of the student.

3. <u>Translations are demonstrated via a key word match</u>. In 2 above, an example was given of a translation that can be called a "key word match." In a key word match,

key words in the English sentence are directly mapped onto symbols in the algebraic expression. The order of the words is preserved in the symbols, as seen with the previously cited example.

$$\underbrace{\text{First number}}_{x}\ \underbrace{\text{is equal to}}_{=}\ \underbrace{\text{the second number}}_{y}\ \underbrace{\text{decreased}}_{-}\ \text{by}\ \underbrace{12}_{12}$$

At least 12 of the texts used this method to demonstrate translations.

The inadequacy of this technique is made obvious when one tries to apply it to the Students and Professors problem. In the sentence, "there are six times as many students as professors," the verb "are" does not come between the "students" and the "professors." As a result, subjects solving this problem often erroneously associate the equal sign with "as." They also use S and P to be the labels for student and professor, respectively. This results in the reversed equation, 6S = P, as shown below.

$$\text{There are}\ \underbrace{\text{six}}_{6}\ \underbrace{\text{times as many}}_{x}\ \underbrace{\text{students}}_{S}\ \underbrace{\text{as}}_{=}\ \underbrace{\text{professors.}}_{P}$$

The teaching of a direct mapping from words to symbols is an attempt at an algorithm for translations. When it works, (and those texts who use this method usually only present problems for which this method would work), the student can perform the problem passively. Column 4 in

Tables 8 and 9 of Appendix D is marked if this translation technique was used in the text.

Confusing or Ambiguous Presentation

Is the presentation of the concept of variable confusing and/or ambiguous and if so, how? Conservatively, at least 25 of the 41 texts reviewed can be described as being, in some way, ambiguous or confusing concerning their presentation of the concept of variable. (The numbers below refer to columns in Tables 6 and 7 of Appendix D.)

9. *Contradictory usage of symbols.* Page (1964), in *Number Lines, Functions and Fundamental Topics*, describes a function, b, that maps the "frame" $\square$ (a placeholder for a number) to the expression $\square + 5$. He writes $\square \xrightarrow{b} \square + 5$. (This symbolization in itself might be confusing to a beginning algebra student.) On the very next page, he introduces the symbol, b, as meaning "back" on a number line. That is, b2 means go back two on the line. Eventually he shows that b2 = -2. The two adjoining pages contain two completely different usages for the same symbol.

Another example of potentially confusing symbolism is the following. Throughout students' science curricula, the letter g is used as a label that stands for the word gram. Bolster, et al (1978) in *Mathematics Around Us: Skills and Applications*, however, use g in some problems to stand for the number of grams. (p.26) This contradictory usage could conceivably result in the following scenario. A student who believes g stands for the word gram, when asked to write an equation that shows there are .0022 pounds per gram, might write .0022p = g where p means pounds. Another student, after reading Bolster,s text might write .0022g = p where p is the number of pounds and g is the number of grams. Because the letter g is most commonly used as a label for the word grams, Bolster may be contributing to students' confusion by, without alerting the students to any switch, letting g stand as a variable for the number of grams.

10. Overabundant use of letters for symbols. Dolciani, et al (1977), in Modern School Mathematics: Structure and Method, use letters on one page of the text to stand for the name of a set. On a following page, letters are introduced as variables standing for elements of that set. (pp.39,41) Though the authors do distinguish between the two uses by using upper case letters in the former example and lower case in the latter, it is unclear whether the students are made to be attuned to that distinction. Similarly, Dolciani and Wooten (1970) in the High School text, Modern Algebra: Structure and Method, use letters in some places as names for points on a number line and as variables in other places. (pp.19,32) Beberman uses symbols copiously, including logical quantifiers like

$\forall x$ (for every x) and expressions like:

$$P = [(x, y), x \in \mathcal{S}_{1 + w} : y = 2] \cdot (1 + w) \quad \text{(p.104)}$$

Though there is no formal contradiction in the multiple symbolic usage of letter in these texts, one must consider the possibility that it contributes to the confusion students have surrounding the concept of variable.

11. Lack of definition or adequate discussion. As mentioned earlier, Küchemann (1981) describes several texts that present the concept of variable either as an analogy to an object (2a + 3a = 5a is like two apples plus three apples equalling five apples) or worse, as the object itself (a = apple). In Davis' text (1964,p.27) that was referred to earlier, he calls the symbol for a placeholder, , a "thing" and says that is therefore two and one half "things." The other forty texts reviewed here are not quite so blatant in associating letters or variables with objects. But neither are most of them explicit in warning the students about the fallaciousness of doing so. Most texts spend very little space and time discussing the concept of variable. The traditional texts devote at most a page to explaining the concept but usually much less. Some do not even define the concept at all. This is true despite the fact that High School Algebra is predicated on the existence of variables, and that x's predominate on virtually every page of the texts.

These texts that skirt the issue of defining and discussing letter use in algebra can be contrasted with the Stanford University School Mathematics Study Group's (SMSG) Programmed First Course in Algebra, written by Creighton, et al (1964). In it are exercises like the following:

> Let us consider the open phrase 7w. Which of the following meanings might w and 7w have in the problem?
>
> A. w is the water in the well and 7w is seven times the water in the well.
>
> B. w is the number of wolves in a zoo, and 7w is the number of wolves in seven zoos.
>
> C. w is the number of dollars I paid for one bushel of wheat and 7w is the number of dollars I paid for seven bushels of wheat.
>
> Answer
>
> A. Don't you mean the number of gallons of water or the number of pounds of water? Remember that variables must always represent a number...
>
> B. This is a possible choice, but would be a correct one only if you knew that each zoo had the same number of wolves.
>
> C. You are correct... It is more reasonable to expect each of several bushels of wheat to cost the same amount than to expect each of several zoos to contain the same number of wolves. (p.141)

Throughout its instruction on word problems, this text continually reminds the student that the letters stand for numbers, and forces the student to actively assign a meaning to the letters.

Tables 6 and 7 of Appendix D indicate which texts fall under columns 8, 9 and/or 10 described above.

As can be seen in Appendix D, the styles of presentation and the quality of usage of letters varies enormously from text to text. It is not within the scope of this paper to either rate the texts in terms of their presentation of the uses of letters or to compare the behavior of students who use different types of texts.

What this dissertation will attempt to do in the next two chapters is to assess students' understanding of one facet of letter use in algebra, specifically, student conceptions of semantically laden letters. In Chapter VI, some of the implications of that assessment vis-a-vis textbooks and curriculum will be discussed.

CHAPTER IV

PRELIMINARY RESEARCH: A FOCUS ON STUDENTS' UNDERSTANDING OF SEMANTICALLY LADEN LETTERS

In chapter II, five questions were presented that can be used to determine "matter meant" vis-a-vis the use of letters in algebra. In chapter III those five questions provide the framework of a textbook review that attempted to describe "matter taught," i.e., how the various texts presented the use of letters in algebra.

Those same five questions could also provide a framework for investigating "matter learned," i.e., how students understand letter use in algebra. How do students understand letters in terms of the types of domains they can have? Is there a difference in students' conceptions of letters found in one dimensional expressions from those of letters found in two or more dimensions? Do students view letters as varying discretely, continuously, or not at all? Do students make a distinction between solution sets and domains? Finally, what are student conceptions vis-a-vis semantically laden letters?

Thus, the five questions provide at least five different research questions that could be investigated. To do so would be beyond the scope of this dissertation. What follows then is a focus on one of those topics, the fifth one. This topic, student conceptions of semantically laden letters, is particularly pertinent to the current focus of many math educators. Problem solving has gained prominance in math education circles, as evidenced by the fact that the National Council of Teachers of Mathematics made problem solving the focus of their 1980 yearbook (Krulik and Reys, ed.s,1980) .and its number one recommendation for the 1980,s in An Agenda for Action (1980, pp.1-5).

It is in the kind of problem solving that applies algebra to word problems and real life situations where meaning is forced on to the letters being used, complicating the symbolization process. A student must not only recognize that letters stand for numbers but must also keep track of the added semantic meaning associatied with the letters. Now, x must not only be thought of as a number, but also as a number or amount of some quantitative entity to which it refers. One can say that the letter now carries a semantic load. It is these letters that have been referred to above as semantically laden letters.

For this author, the pursuit of an understanding of student conceptions of semantically laden letters was motivated by the previously cited works of Clement (1981), Clement, Lochhead, and Monk (1981), Rosnick and Clement (1980), and Kaput and Clement (1980). Specifically, the finding that students often used letters to stand for labels rather than for numbers provided a working hypothesis for preliminary stages of the investigation. (The previously cited findings of Galvin and Bell (1977), Matz (1979), and Küchemann (1981) also support this working hypothesis.)

The preliminary research can be broken into four specific stages. They are as follows:

1. Clinical interview transcripts of students solving the previously described Students and Professors and Cheesecake and Strudel problems were reviewed in search of additional evidence of the "labels" approach to letters in algebra.

2. A multiple choice version of the Students and Professors problem (described below) was given as a written diagnostic test to further document the "labels" phenomenon.

3. The Books and Records problem (described below) was given as a written diagnostic test.

4. Clinical interviews on the Books and Records problem were held to further understand the results of the written tests.

The following is a description of some of the results of the preliminary research.

Initial Clinical Interviews

Some of the strongest data from which it has been inferred that students view letters as labels rather than numbers came from clinical interviews. College students, most of whom had had at least one semester of calculus were interviewed on problems similar to the Students and Professors and Cheesecake and Strudel problems. One student, a Statistics student (who had had one semester of calculus) claimed that the equation 4C = 5S represents the fact that at a particular restaurant, for every four cheesecakes that were sold, five strudels were sold. (In this problem the student was told that C stands for the number of cheesecakes sold.) When asked whether the C could be replaced with a number, she responded as follows, "I'm saying that C equals cheesecake and C can't really equal a number because cheesecake is cheesecake. Cheesecake isn't a number. four equals the number of cheesecakes, It [the C] is just explaining what four is multiplying."

Another student, also taking a college statistics course, described the meaning of the expression 6x as follows. (Its correct meaning in context is 6 dollars times x, the number of turtles. The problem he is working on is problem 1 of Appendix E.)

S: ...6 means 6 dollars. The amount that the turtles cost. And x means turtles.

I: That's still confusing me when you say x stands for turtles.

> S: It's like, you know how you use n or something in a math equation or x or whatever, umm, I could just write 6 turtles here instead of the x. (The student writes "turtles" over the letter x.) It's a shorter way of writing it.

Notice that though initially this student recognizes that the six stands for the price of one turtle, when read in combination with the x, it becomes an adjective for the noun, turtles. Instead of six dollars times some unknown number of turtles, she now has six turtles. x is just a shorter way of writing the word turtles.

These data, which tended to support the working hypothesis, prompted the development of the problem given in the second stage of the premilinary research.

<u>Diagnostic Test: A Modified Students and Professors Problem</u>

This problem, a version of the Students and Professors problem, was given to 33 sophomore and junior business majors in my statistics course. Most of these students had had two semesters of calculus. It was also given to 119 students in a second-semester calculus course designed for the social sciences. The problem reads as follows:

Students and Professors Problem II

At this university, there are six times as many students as professors. This fact is represented by the equation S = 6P.

A) In this equation,
what does the letter P stand for?
i) Professors
ii) Professor
iii) Number of professors
iv) None of the above
v) More than one of the above
(if so, indicate which ones)
vi) Don't know

B) What does the letter S stand for?
i) Professor
ii) Student
iii) Students
iv) Number of students
v) None of the above
vi) More than one of the above
(if so, indicate which ones)
vii) Don't know

The results were more startling than expected and, again, supported the working hypothesis. Over 40 percent of the 152 students were incapable of picking "number of professors" as the only appropriate answer in Part A. Similarly, over 43 percent did not answer part B correctly. Even more compelling is the result that 34 people (over 22 percent) chose as their answer "S stands for professor." It is important to note that every person who chose "professor" for the answer in part B chose "none of the above" for their answer in part A. The latter is a consistent response in that those who would view S as standing for "professor" would also view P as standing for "student." Since that option was not provided, they chose "none of the above."

These results seem to support the hypothesis that students tend to view the use of letters in equations as labels that refer to concrete entities. S stands for "student" or "students" or "professor," not the more abstract "number of students." (These preliminary data were reported in Rosnick (1981), an unabridged version of which is included as Appendix B.)

Diagnostic Test: Books and Records Problem

The following problem (example 2 in Appendix E) was developed to see whether, when faced with a blatant contradiction, students would recognize the fallacy of a labels reading of a letter. The problem reads as follows:

> I went to the store and bought the same number of books as records. Books cost 2 dollars each and records cost 6 dollars each. I spent 40 dollars altogether. Assuming that the equation 2B + 6R = 40 is correct, what is wrong, if anything, with the following reasoning. Be as detailed as possible.
>
> $2B + 6R = 40$
> Since B = R, I can write:
> $2B + 6B = 40$
> $8B = 40$
>
> This last equation says eight books is equal to 40 dollars. So one book costs 5 dollars.

It was thought that if students view letters as standing for labels, they would agree that 8B reads as "eight books" rather than "eight times B, the number of books bought."

Thirty statistics students , all of whom had had two semesters of calculus, were given this problem as a written diagnostic test. Of those thirty, 23 (77%) did not recognize the misinterpretation of the last line, giving some erroneous explanation of how the problem was done incorrectly. Furthermore, only three of nine junior and senior math majors correctly solved this problem. At first glance, these data again seemed to support the working hypothesis in that a large majority of the students did not see the error of interpreting 8B as "eight books." However, closer inspection of student responses suggested that the working hypothesis was not sufficient in describing these students' conceptions. By far, the most common incorrect answer to the above problem was that even though the number of books bought was the same as the number of records bought, B does not equal R because their prices are different. Thus, the letter B seemed to stand for more than just the word "books"; it also stood simultaneously for the price and number of the books. To get a better sense of what students were thinking, clinical interviews of students solving this problem were conducted.

Clinical Interviews: Books and Records Problem

These interviews seemed to confirm that students allowed letters to take on more than one meaning, seemingly simultaneously. One woman (Ann, a student in a college calculus course), in the course of a thirty minute video tape interview on the problem described below, gave what seemed to be at least seven distinct different interpretations of what one of the letters used had meant. What was intriguing about this interview was that Ann gave no indication whatsoever that these different interpretations were either inconsistent or contradictory. It should be noted at this point that the following transcript sections are described by looking at each statement Ann makes and each equation she writes in isolation. Comments on the statements and equations are meant to be descriptive, not interpretive or judgmental. It should be reemphasized that though these snippets of transcript are described above as being distinct interpretations, no claim is being made that Ann is aware of those distinctions. In other words, it is possible that these statements and equations that appear to the reader to have separate meaning might, for Ann, be a part of some larger amorphous conception of what the letter means. (The next chapter will present evidence which strongly suggests that is, in fact, the case.)

The following are excerpts from the transcript of Ann solving the problem. She begins by reading from the problem.

S: Ok, I went to the store and bought the same number of books as records. So, use B for books...

Here the letter is first presented with a very generalized meaning. There is no indication of which attributes, if any, are associated with B nor is there any indication that B is even quantitative.

> S: (reads) "What is wrong with the following reasoning?"...Um, they say that "since B equals R, I can write" -; But B doesn't equal R because they said that B equals um,..is the number of books which is 2 dollars each and R is the number of records at 6 dollars each.

Here the letter seems to represent a combination of quantitative attributes, mixing a quantitative attribute of a set of objects (the number of books), with a quantitative attribute of individual objects (the price of each book). Note that Ann seems not to be troubled with somehow associating the letter B with the constant 2. Ann continues:

> S: You have to treat them as separate variables. You can't add different variables together.

> I: So tell me again why you say B does not equal R?

> S: B doesn't equal R because B is-is books, which are 2 dollars each and R records which are 6 dollars each...

Ann's explanation has changed subtly. She no longer talks about the number of books but about the generalized object, books. B now is a letter that represents a combination of generalized object and a quantitative attribute. Note again that numerically, B is connected with the constant 2.

Next, Ann analyses the equation 2B + 6R = 40 and reads the 2B as "two books." Here B appears to be a <u>label</u> for the word books. Later, she plugs a two in for the B, saying "2 books times 2 dollars is 4 dollars." Now B seems to stand for the <u>quantitative attribute of an individual object</u>, the price of a book. Still later, when pressed to describe what B means "in as detailed way as possible," Ann says:

> S: Well, B is the number of books, but it -; more importantly-- in terms of figuring out how much they cost--B is -is a price, which is 2 dollars.

Again, Ann attributes a <u>combination of quantitative attributes</u> to the letter B. Later:

> S: Well, B is one book because it-it; B has something to do with the price of the book--B would be the um...the...the price of 1 book.

This statement appears to be subtly but importantly different from when she said "B is books which are two dollars." The quantitative attribute of the object (the price) is no longer associated with books in general but with a singular prototypical book. There is a sense now that B stands for an individual <u>object</u>, a book that has a price.

Ann continues to work the problem from several angles. At one point, she was asked to rework the problem answering the specific question, how many books were bought. She was able, after some time, to answer that question by means of a trial and error arithmetic solution. But in trying to relate her solution to the algebra, she again revealed confusion concerning the meaning and role of the letters. She first wrote 40 = B + R saying "the total amount was $40 equals some number of books plus some number of records." Verbally, Ann defines the variable as number of books but in order for that equation to be accurate, B would stand for the total amount of money spent on books. She then replaces the B with a 2, the R with the 6 and says:

> S: I added them together which gave me 8 dollars. And I-I add B and R but I knew that when I divided it, it was just going to give me the the number of books or the number of records...um, so I just- crossed out the R. That would give me numbers of books (crosses out the R and gets 40 = B + R).
>
> I: How did you get that "B" there again?
>
> S: Well, um...I...I was;..figuring out how many books, so I needed a, um B to be the number of books.

B is originally replaced by 2, implying that it means the price of a book, after which some manipulations were done (dividing 40 by 8 and solving for what at that point is an invisible unknown!). Finally, B ends up standing for the number of books. B here seems to have a <u>shifting</u> <u>meaning</u> from one quantitative attribute to another.

Still later, Ann obtains the correct equation, 40 = 5(2) + 5(6) and maintains it means the same thing as did 40 = B + R.

> I: When I point to the B here [in 40 = B + R], would you point to what is the B in this equation? [40 = 5(2) + 5(6)].
>
> S: B is the um...the number of books at a-at a

certain price; at the price that was \$2.00.

I: So-so where would it be here? [points to 40 = 5(2) + 5(6)]

S: So it would be this right here. B is this whole (circles with her finger the term 5(2)).

According to Ann, B represents the total price spent on books which is 5(2). Finally, as if to underscore her own confusion, when asked if B in any way could equal 5, Ann says, "I think B is equal to 1..but, um. it-I think you're referring to it right here...where you could say B is equal to 5-."

Ann's transcript and those of other students who were interviewed in this preliminary phase of the research dramatically reveal deep confusion about the meaning and use of semantically laden letters. The meaning that these students seem to impart to these letters is tenuous, vague, unstable, and only sporadically quantitative. In the next chapter, a new hypothesis for explaining a prevalent student conception of semantically laden letters is formulated. Both clinical and written diagnostic data is presented to describe and document this phenomenon.

CHAPTER V

A REDEFINED HYPOTHESIS AND SUPPORTING EVIDENCE

The preliminary results reported on in the last chapter made it apparent that a description of the labels approach was necessary but by no means sufficient in explaining what these students were thinking. Though students used the letter to label the referent, they at other times replaced the letter with a number or in some other way recognized the quantitative nature of the letter. It became apparent that a new hypothesis was needed to help explain these student behaviors. This motivated four stages of research that followed the following sequence:

1. Clinical Interviews.

2. Exploratory Analysis of Transcripts: the creation of a new hypothesis.

3. Refined Analysis of Transcripts: the selection and description of behavioral criteria that support the hypothesis.

4. Diagnostic Tests: further evidence in support of the hypothesis.

Clinical Interviews.

The clinical interviews were designed to further investigate student conceptions that were revealed in the preliminary studies. Several word problems were given to nine students who were interviewed as they attempted to solve these problems. These interviews were recorded on audio tapes and then analyzed. Different problems were tried (see Appendices E and F) but all had several things in common. They all could be solved by setting up two equations with two unknowns. The entities involved in the problems all had two attributes of consideration, e.g. price and quantity or weight and quantity. In each case, one of those attributes was given and the other was left unknown. All problems had integral solutions. The following is an example (example 1 in Appendix E).

> A boy bought a collection of frogs and turtles. The number of frogs that he bought was 3 times the number of turtles that he bought. Frogs cost 3 dollars each and turtles cost 6 dollars each. He spent 60 dollars altogether. How many frogs did he buy?

It was believed that this type of problem would be favorable in determining student conceptions of semantically laden letters because having two relevant attributes for each referent makes the differentiation between those attributes more crucial to the successful completion of the problem.

Eight of the subjects were chosen at random from among a group of volunteers who were all enrolled in a statistics course designed for the social sciences at a large university. The ninth was similarly selected from a college calculus course. Seven of the nine had had or were currently enrolled in a college calculus course. Two of those seven students had taken two semesters of Calculus. The subjects were paid a nominal sum for their time. Subjects were told that the purpose of the interviews was to learn more about how students solve these types of problems and for that reason they should talk out loud as much as possible.

Interventions on the part of the interviewer were limited to questions that asked for clarification, amplification, or identification, such as "what did you mean by that?," "could you say a little more about that?," "could you read what this equation means in English?," etc. Other questions aimed at students' understanding of letters, like "what does the x mean in this expression?." The interviewer was conscious of the pitfalls of asking leading questions and avoided doing so as much as possible. Some of the basic interviewing guidelines that were followed are given by Konold and Well (1981), and by Pines, et al (1978). Each student was interviewed for approximately 45 minutes in which time they solved an average of just over three of the word problems of the type given above.

Toward a New Hypothesis: The Exploratory Analysis of Interview Transcripts.

The tapes of the interviews were transcribed and studied in search of common or idiosyncratic patterns or behaviors on the part of the students. Several patterns became apparent. Several students attempted to solve the problems, often awkwardly, with trial and error arithmetic methods. Other students avoided functional relationships that expressed one variable in terms of another, using equations in one variable only. Some students expressed frustration with word problems in general. But all students demonstrated disturbing misconceptions and attitudes towards the use of letters in solving these problems. One of the many indicators of that is the fact that not one of the students spontaneously and correctly identified the quantitative meaning of the letters they used, either orally or in written form.

The students' transcripts were reviewed in small, isolated segments consisting of verbal statements or written equations. As had been the case with Ann, it appeared as though letters were often being used in a very inconsistent manner. Also, as in the case of Ann, students gave little indication that they were aware of that inconsistency.

<u>A categorization of observed student uses for letters</u>.

The following is an attempt at categorizing the types of letter use observed in those isolated segments. Again, this list is meant to be descriptive of student behavior, not judgmental or interpretive of cognitive processes. It is an attempt to classify statements and written equations into categories of behavior on the basis of either a literal interpretation of the words in the statement or a mathematical interpretation of what the letters would mean if the written equation were true. This classification scheme should not be compared with that of Küchemann presented on page 29 because of its noninterpretive nature and because it deals solely with semantically laden letters that are found in the type of word problems that comprise this study.

The phrases in parentheses following the first eight categorizations below refer to examples from Ann's transcript presented above.

I. <u>Generalized letter</u>: the letter is verbally defined as generalized, nonquantitative entities. (B is books.)

II. <u>Letter as generic label</u>: an expression consisting of a number followed by a letter is read as if the number were an adjective modifying the letter. The letter is a nominal label for the referent. (2B is read as two books.)

III. Letter as object: the letter is verbally defined as a single, nonquantitative entity. (B is a book.)

IV. Letter equaling 1: a letter is defined as in III above and given the value of 1. (B is equal to 1 because (ostensibly) it refers to one book.)

V. Letter as quantitative value of an attribute of an object: (B is the price of one book).

VI. Letter as quantitative value of an attribute of a set of objects: (B is the number of books in the set of books bought or B is the total price paid for all of the books).

VII. Letter as a combination of quantitative attributes: (B is the number of books and also the price of the books).

VIII. Letter as a combination of qualitative and quantitative attributes: (B is the books at a certain price).

IX. Letter as serially representing each of the items in a set: This is only subtly different from letter as generic label, in that in the labels approach, to represent 3 frogs, one writes 3X, whereas here 3 frogs is written as 3X's. Furthermore, one has a sense that each x represents one of the frogs being discussed, not just any frog. One student said the following, in partially explaining her reasoning for the equation $6y + 3(3x) = 60$:

S: I think the y was the turtle and the x was the frog. For every 3 frogs you have 1y. So for every 3x's you have 1y...If a y cost $6, um, I somehow figured if you took y and multiplied it by 6 and then you had 3 er, the 3x's times 3 which actually sort of makes sense..you have $6 for every y and...$9 for every x.

Notice the dual roles of the letters. The y for example is a number that is multiplied by 6 but it is also a word, vis. "$6 for every y."

X. Letter as representing some abstract quantitative attribute of the referent. (An example of this is found in a situation where turtles are bought in a 3 to 1 ratio to frogs. Here T might be identified with the 3. Three is not the number of turtles bought nor is it any other quantitative attribute of turtles other than the integer representing the ratio associated with turtles.)

XI. Letter as a combination of attributes from different referents. One student said the following:

S: When I was doing this equation right here, [4(1x) + (3x) = 91], I said that, umm, 4 times 1x and x would be the red block and plus 3 times x, and x would be the green blocks too, and I wanted to do, and the weight of a green block is say x and we have 3 of them so it's going to be 3x and the weight of the red block is x and we have 1 of them, but the weight of a red block is actually 4 times x because the weight of the red block is 4 times as big as the weight of the green blocks.

The x seems to be simultaneously representing attributes of both red and green blocks.

It is important to emphasize that though categories I through VIII are a result of a post facto analysis of Ann's transcript, they do not appear to be separated conceptions in Ann's mind. For Ann, these eight categories seem to be part of a more global, amorphous, less differentiated conception of what the letter means and how it is used. This observation can be generalized by saying that many college level math students seem to view semantically laden letters in a manner similar to Ann and that more articulated conceptions (exemplified by some of the individual categories) seem to be a part of that more global view of the letter.

The hypothesis.

This suggests the following hypothesis. For many students, the referent of a semantically laden letter appears to be an undifferentiated conglomerate. That is, the letter is identified with an entire, complex, overly generalized referent itself rather than a particular quantitative attribute of that referant. Thus, though the correct solution of a problem calls for creating a variable, x, that stands for the number of turtles bought, students will use x to mean some vague and general concept having to do with several aspects of turtles. Within this undifferentiated conception of what x means could be one or more quantitative attributes of turtles, or the name turtles, or the actual physical thing called a turtle, and/or etc. In a sense, students allow x to stand for "turtleness"; i.e. it encompasses much of what, in the context of the problem, the word turtle or turtles implies. This hypothesis, that students associate semantically laden letters with an undifferentiated conglomerate is central to the remainder of this dissertation, and, for that reason, bears more description and definition.

It should be emphasized at the outset that the various attributes comprising the conglomerate that students associate with a semantically laden letter are undifferentiated but are not necessarily undifferentiable. A firm analogy may not be made with the difficulties students have in differentiating mass from weight or heat from temperature where the differences are abstractly difficult to pin down. Surely one would expect that college students, when pressed, could differentiate between price and quantity. However, in symbolizing those various attributes with only one letter, students seem to not attend to that differentiation.

A parable taken from Thomas Kuhn's "Second Thoughts on Paradigms" (Kuhn, 1974), may be helpful in further illustrating the above phenomenon. In this parable, a father and Johnny, his very young son, are walking by a pond in which are swimming several geese, ducks, and swans. Kuhn writes:

> Father points to a bird, saying, "Look, Johnny, there's a swan." A short time later, Johnny himself points to a bird, saying, "Daddy, another swan." He has not yet, however, learned what swans are and must be corrected: "No Johnny, that's a goose." Johnny's next identification of a swan proves to be correct, but his next 'goose' is, in fact, a duck, and he is again set straight. After a few more such encounters, however, each with its appropriate correction or reinforcement, Johhny's ability to identify these waterfowl is as great as his father's. (p.473)

At the point in the above story where Johnny has first learned the word swan, he applies that word in <u>an overly general way</u> to refer to all three types of waterfowl that he sees. That is, the word 'swan' for Johnny has an overly generalized referent, just as the letter B for Ann had an overly generalized referent, namely books and all of its accompanying attributes. The word "undifferentiated" is appropriate in that Johnny does not initially attend to the differences between swans and geese even though, as the end of the story asserts, he is perfectly capable of doing so. For Johnny, swans and geese are at first undifferentiated, even though they are differentiable.

This parable, though not completely analogous, helps to illustrate what seems to happen when students view semantically laden letters as undifferentiated conglomerates. In addition, it explicates what is meant by "undifferentiated." In what follows, a similar justification will be given for the use of the word "conglomerate."

The word "conglomerate" in one of its common applications refers to a corporation that is made up of many subdivisions. Gulf and Western, for example, is a conglomerate consisting of, among many others, divisions that grow tobacco in Massachusetts, sugar in the Caribbean, and produce films in California. In the same way, "books" can be thought of as a conglomerate if it is a catch-all for the several attributes and manifestations of books found in a particular problem context.

Furthermore, if one were discussing the profitability of growing tobacco in Massachusetts, one would most likely want to differentiate between the many divisions of Gulf and Western and focus on the one division of relevance, Consolidated Cigar. Speaking of Gulf and Western when one really means Consolidated Cigar, is speaking at an inappropriately generalized, undifferentiated level. In the same way, using "books" as the referent of a letter rather than "number of books," is referencing at an overly generalized, undifferentiated level.

Having thus defined the term undifferentiated conglomerate, the stage was set for the third stage of the research.

Selection and Description of Behavioral Criteria: A Refined Analysis of Interview Transcripts.

With the establishment of the hypothesis, that many students tend to view semantically laden letters as undifferentiated conglomerates,two goals evolved. One was to attempt to establish some behavioral criteria for ascertaining the existence of the conception in a particular student. The other was to begin to determine just how pervasive the conception is among students in general.

Due to the subjectivity inherent in any analysis of clinical interview data, it was felt that the criteria should be strong and convincing. It was felt that simply witnessing students saying "B is books" or "x is turtles" was not strong enough. Though a student saying "B is books" might imply a lack of careful differentiation, it could be argued that the student understands the meaning and function of the letters but just has not learned (or has no reason) to articulate that. It could be further argued that if the student is capable of treating the letters quantitatively then his or her verbalizations are more or less irrelevant.

In a similar fashion, any one piece of evidence pointing to the existence of the conception that semantically laden letters refer to undifferentiated conglomerates could be circumstancially negated. However, if more than one piece of evidence were found, and if those pieces of evidence indicated in different ways that the conception is held by the student, the argument that is made becomes significantly more convincing.

Four behaviors.

The following four behaviors were each observed in several of the students' transcripts. Each one is considered by the author to be supporting evidence for the stated hypothesis. It was felt then that if any student, in a clinical interview, revealed three of these four behavioral criteria, one could conclude that that student tends to identify semantically laden letters with undifferentiated conglomerates.

The determination that three of four behavioral criteria validate the hypothesis is an arbitrary one. It is believed by the author that this standard for testing the hypothesis is sufficient but not necessary. That is, if less than three of the behaviors are observed, it does not necessarily imply that a student does not have that misconception. For that reason, this standard is considered to be a conservative one. The four behaviors are as follows:

A. The student does not give clear, stable, quantitative definitions for the letters he or she uses.

B. The student evidences a use of letters that would fit either category VII, VIII, or XI above, and/or interprets letters in a way consistent with more than one of the remaining categories, indicating multiple or shifting meanings for the letter.

C. the student accepts the juxtaposition of two contradictory quantitative uses of the same letter.

D. The student is unable to drop part of the semantic load that a letter carries even when it is necessary and appropriate to do so.

To illustrate each of these behaviors, edited transcripts of several of the students will be presented and analyzed. It should be noted that although these transcripts have been selected to illustrate, individually, behaviors A-D, each transcript is rich with other behaviors vis-a-vis the use of semantically laden letters.

Behavior A. The omission of a clear, stable, quantitative definition for the letters being used was evidenced by all nine students on at least some of the problems they tried and by eight of the students on all of the problems they tried. In the following transcript segment, note that Eileen manages to solve the problem without having a clear understanding of what the letters she uses mean. Eileen had previously taken two semesters of a rigorous calculus course and one semester of physics, and was currently enrolled in a statistics course. (Numbered notes and comments will follow the transcript segment. The problem is given in full as example 3 in Appendix E.)

081 S: (Reads) "A person went to the store and bought pecans and cashews and got a total of 100 nuts." So I'm going to put to start out with; P for pecans plus C for cashews is equal to 100. (Writes P + C = 100) [1]. (Reads) "The number of pounds of

pecans he bought was the same as the number of pounds of cashews. 8 pecans were 1 pound and 12 cashews weighed 1 pound. How many pounds of pecans did he buy?" So I'm looking at a total of 100 nuts and 8 pecans weigh 1 pound so 8 nuts, 8 of these weighs 1 pound, 12 of these weigh 1 pound and so I would just put that into proportion, I guess. Umm, I;m going to do the same thing I did with turtles and frogs [2]. I'm going to say that P is equal to pecans [3] and then, well actually, 8 pecans [4]. So; yeah, 8 is equal to pecans (writes 8P = pecans) and then 12P is equal to cashews (writes 12P = cashews) and then I'm going to say 8P plus 12P is equal to 100 (writes 8P + 12P = 100) [5]. 20P is equal to 100. P is equal to 5 [6]. Umm.

082 I: And what does P stand for?

083 S: P is a number again [7]. So pecans would go 8 times 5 is equal to pecans; and that's 40. And then 12 times 5 is equal to cashews and that's 72? 60. And then to check it out, that's the number of nuts he bought and it should equal 100 because that was given in the beginning. And the question is how many pounds of pecans did he buy? [8]. Well, he bought 40 pecans actually in number so I would take the 40 and, umm, I would take the 40 and divide it by 8 to get 5 because there are 8 nuts in every pound so, or if there is 8 nuts in 1 pound then how many pounds would there be for 40 nuts (writes 8 nuts/1 lb = 40/x lb) [9]. So I'd go, 40 is equal to 8 nuts (writes 40 = 8x) [10] and get x is equal to 5.

Later on the interviewer recapped this final segment.

090 I: You multiplied [the 5] times 8 and got 40. Then you divided 40 by 8 to get 5 again.

091 S: Yeah, So I probably could have stopped but I didn't recognize that that was the answer.

092 I: Why do you think that... you didn't

recognize the answer?

093 S: Because probably not until I looked back did I realize that the P I was trying to find was pounds [11]. If I realized it first then I would have said P is pounds [12] and P is 5 then 5 pounds of pecans is what he bought. But not until you asked me what was P, what was P standing for then I realized that it had to be pounds.

The following numbered comments refer to portions of the above transcript.

1. Eileen verbally identifies the letters as in category I (P is for <u>pecans</u>) but writes an equation that suggests that P is the <u>number of pecans</u> bought (category VI).

2. Eileen had previously worked on the frogs and turtles problem given on page 89 and despite numerous confused starts and a good deal of reversed reasoning (e.g., she said, "the number of turtles he bought was 3 times the number of frogs so T plus 3T is equal to the number of turtles and frogs he bought"), she correctly solved the problem.

3. Again this is evidence of category I.

4. This is possibly category X. P is associated with 8 pecans and in fact, if taken in isolation, this sentence says P = 8. Eight is an abstract quantitative attribute of pecans (8 pecans per pound).

5. The last verbalized definitions of P were that P is equal to pecans and that P is equal to 8. But in order for this equation to be correct, P has to be the number of

pounds of pecans sold (category VI). In any case, note the unacknowledged shift of meaning and utilization of P from P + C = 100 to 8P + 12P = 100. This is an example of behavior C.

6. This is the correct answer, but, as will be seen, Eileen doesn't yet recognize it as such.

7. She recognizes that P is quantitative but gives no clear definition of its meaning.

8. She does not know! This supports the claim that when she wrote 8P + 12 P = 100, she was not thinking that P meant the number of pounds of pecans bought.

9. Note the introduction of the new variable x. This again supports the above claim.

10. Note the "labels" reading of the letter (category II). 40 is equal to 8 nuts where nuts is said as she writes the x.

11. This seems to be an affirmation of the fact that as she solved this problem, Eileen did not have a clear, stable, quantitative definition of what P meant (behavior A).

12. Note now that "P is pounds" is consistent language with "P is pecans"; it still fits, at least verbally if not cognitively, into category I.

Thus, though Eileen was able to solve this problem, she did so without a solid grasp of the meaning of the letters that she used. Furthermore, what descriptions of the meanings of the letters she had, shifted from category I to category VI, to category X, to a different manifestation of category VI. This transcript therefore exemplifies behavior B as well as behaviors A and C.

<u>Behavior</u> <u>B</u>. The following transcript segment (of a student solving example 4 in Appendix E) gives another example of a student performing behavior B. Note that though this problem is similar to the previously described Books and Records problems, it differs in two significant ways. First, in this problem the number of books and records bought is known and the prices are unknown, whereas in the original problem this was reversed. Secondly, this problem has not been done out for the student. Paul has had one semester of calculus and was currently taking a statistics course. He begins by reading the problem.

001 S: "A person went shopping for books and records. He spent a total of $72. The price of each book was the same as the price of each record. He bought 2 books and 6 records. What was the price of 1 record?"---He spent $72--He bought 2 books and 6 records. What was the price of one record? The price of each book was the same as the price of each record. So x is gonna be for... x equals x [1] (mumbles to self. Writes 72 = 2x + 6x; 72 = 8x; 9 = x [2]

002 I: What are you uh, remarking about?

003 S: Uh-- it just didn't look right for a minute.

004 I: Uh-huh.

005 S: What was the price of 1 record--(to self) The price of each book was the same as the

price of each record. He bought 2 books and 6 records for $72--I'm getting confused here.

006 I: What's confusing?

007 S: --How do we write this answer? To-to find the price of the record. I suppose it's 6 records.

008 I: Ok. When you said "6 records" you underlined 6x (in 72 = 2x + 6x). Is that it?

009 S: Right [3].

010 I: Ok--Can I ask you what x stands for, just so I can be thinking with you?

011 S: Um,--(Pauses) [4].

012 I: What were you thinking?

013 S: Items purchased [5].

014 I: Items purchased? Uh-huh. So, read this first line to me again. [72 = 2x + 6x]

015 S: $72 equals 2 items plus 6 items [6].

016 I: Ok--And what is it that you're troubled by now?

017 S: It doesn't seem to be answering the question.

018 I: How's that?

019 S: Uh---

020 I: When you say 9 is equal to x, what do you conclude from that?

021 S: I don't know [7].

022 I: Er, x means? What does x mean?

023 S: A minute ago I said x means items purchased.

024 I: Uh-huh.

025 S: 9 items purchased.

026 I: Uh-huh. Does that seem reasonable to you?

027 S: Yeah.

028 I: Ok.

029 S: ---(30 seconds)---No.

030 I: What was that?

031 S: I said no, it doesn't seem reasonable.

032 I: Oh.

033 S: 8 items purchased.

034 I: Uh-huh. You pointed to the 8x when you said that.

035 S: Yeah. X is items purchased. This is all very confusing [8].

036 I: Uh-huh. It's a confusing problem.

037 S: So I guess uh,--it's $9 for 1 record.

038 I: How did you decide that? Now what you just did was, you put dollar signs on the 9 and the 72 and you just said $9 for each record. How did you come to that conclusion?

039 S: There was 8 items purchased.

040 I: Uh-huh.

041 S: For a total of $72. And the price of each book was the same as the price of each record. So everything costs $9.

042 I: I see. Um, so you say 8 items were purchased when you look at which equation?

043 S: $72 = 8x; 8 items.

044 I: Uh-huh. So x means what?

045 S: Items purchased.

046 I: Items purchased. And then you have x is equal to $9.

047 S: Right. Items purchased equals $9; those were the prices. $9 per item purchased [9].

048 I: It sounds like you have a question in your voice when you're saying that.

049 S: Uh, each item costs $9.

050 I: Uh-huh; So say again now specifically what x means.

051 S: The cost for an item [10].

052 I: The cost for an item. Did you, er, were you thinking that when you first wrote the equation down? You just said x is the cost of 1 item. Were you thinking that when you first wrote it down?

053 S: I don't think all that consciously, but maybe a little bit subconsciously. I think that's what it had to be; yeah [11].

The following numbered comments refer to portions of the above transcript.

1. It is unclear what x equals at this point. When he says "x equals x" it is possible that that is parallelling the sentence "the price of each book was the same as the price of each record," but that is not clear.

2. This is the correct answer but it seems that he doesn't recognize it as such yet.

3. The implication is that x is a label meaning records (category II).

4. The long pause might indicate that he doesn't clearly know what x means (behavior A).

5. X seems to stand for both records and books so it is

generalized further to stand for items purchased (category I).

6. X is used as a label for the word "items" (category II).

7. This confirms that he doesn't recognize that he has the right answer and therefore seems to confirm that he doesn't recognize that x stands for the price of records (or books).

8. Again, this is a labels interpretation of x.

9. This is an example of category VIII. X is simultaneously "items purchased" and the quantitative value $9.

10. Finally, Paul is able to give a good definition of the letter (category IV).

11. If in fact Paul was thinking correctly subconsciously, it must have been pretty far back in his consciousness because he didn't recognize the right answer when he saw it. A better explanation might be that because the qualitative concept "items purchased" was mixed with the quantitative attribute, price of an item, Paul became confused.

<u>Behavior C</u>. As was the case with Eileen's transcript, the preceeding transcript not only exemplifies in particular one of the behaviors (in this case behavior B) but also gives strong hints of other behaviors (A and C). The following transcript segment focuses on behavior C, but

also includes some of the other behaviors. Behavior C is distinguished by the juxtaposition of two quantitative interpretations of one letter without any remorse or discomfort on the part of the problem solver.

Ray, also a student enrolled in statistics with a calculus background, is working on the same pecans and cashews problem that Eileen worked on (example 3 in Appendix E). After he reads the problem, he writes down the information using a labels approach. Thus, he writes 8P = 1 lb and 12C = 1lb to mean that there are eight pecans and, respectively, twelve cashews to a pound. He quickly follows this by writing P = C, saying "the number of pounds of pecans he bought was the same as the number of pounds of cashews." Note that this is an example of behavior B. P first means the word "pecans" and then means the number of pecans bought. Ray becomes stuck in trying to solve the problem algebraically and resorts to a trial and error, arithmetic solution. He is successful after several attempts. We pick up the transcript at that point.

047 I: So you solved it by trial and error.

048 S: Right. I just plugged in a couple of numbers. I'm sure I could have worked it out in a formula.

049 I: Could you do that for me? I'd be interested in seeing the formula.

050 S: You want a formula. Well, you know P equals C, so...So it's got to be, umm 12C + 8P = 100 and let me see. P equals 3/4C because 8 is...No, 2/3, I'm sorry (writes P = 2/3C). 8 is two thirds of 12,

right [1].

051 I: So P is equal to two thirds C.

052 S: So when you multiply...No, it doesn't work either.

053 I: what were you about to do?

054 S: I was going to multiply 8 times two thirds but it comes out to 16, it comes out to 16 thirds and that's not a very nice figure to work with [2].

055 I: What does this equation mean, 12C plus 8P equals 100. Could you read this in English?

056 S: In English? Umm, well, you know there is 100 nuts and there are 12 cashews to a pound and there is 8 pecans to a pound so you know from the formula there is going to be...the proportion of nuts is going to be um, the total number of nuts is going to be in this proportion [3].

057 I: What does the letter C stand for?

058 S: Cashews. And P stands for pecans [4]. But they're worried about weight, how many pounds of pecans did he buy. You know it's equal to the number of pounds of cashews he bought so, 8 pecans weigh 1 pound and 12 cashews weigh 1 pound. So, you know there is 100 nuts. Let me check this over again. I want to make sure this proportion is right. 12 cashews in a pound and 8 pecans in a pound so 8 over 12 it's got to be two thirds [5].

The following numbered comments refer to portions of the above transcript.

1. Ray has written two equations, 12C + 8P = 100 and P = 2/3C. Both equations are correct but only if you allow the letters to have different quantitative meanings. In the former, P for example, has to stand for the <u>number of</u>

<u>pounds</u> <u>of</u> <u>pecans</u> bought (category VI). In the latter, P has to stand for either the number 8 (category X) or the <u>number</u> <u>of</u> <u>pecans</u> bought (category VI). This exemplifies behavior C.

2. Ray tries to solve the above two equations simultaneously but rejects that, not because the letters mean different things in each case, but merely because the numbers come out to be non integral.

3. the equation here is very loosely defined. It is not clear that for Ray, P and C stand for the number of pounds of pecans and cashews, respectively, nor is it clear that, in Ray's mind, multiplication is occurring between the coefficients and the letters.

4. This hints of category I.

5. Despite acknowledging that the problem is "worried" about weight, Ray does not let go of trying to relate P and C in terms of the numbers of individual pecans and cashews, respectively, in a pound.

Eventually, Ray gives up on the equation, P = 2/3C, only to try the semantically identical equation, 3/2P = C. When that also produces a non integral result, he goes back to rereading the problem. He rediscovers that he can write P = C and solves the problem accordingly. What is important to note is that he never completely rejects the idea of relating P with C via the equation P = 2/3C. The interviewer asks:

067 I: What do you think was wrong with what you were doing over here when you said P was equal to two thirds C?

068 S: Well, I wasn't, I wasn't, umm...I was trying to solve..I wasn't...

070 S: ...What I was doing over here, I was trying to make C and P equal in a proportional sense, you know, by using a fraction, by dividing the 8 into 12 or the 12 over the 8, when...

071 I: What's wrong with that?

072 S: Well, it's not necessary because it's given, it's C equals P...

Thus it seems that for Ray, there is nothing wrong or inconsistent with writing P = 2/3C, it is just that it is not necessary in solving the problem. Ray does not seem to recognize that the meaning of the P is different in P = 2/3C from its meaning in 8P + 12C = 100. This apparent acceptance of the juxtaposition of two contradictory quantitative uses of the same letter is an example of behavior C.

Behavior D. Perhaps the most vivid evidence of a student's conception of semantically laden letters as undifferentiated conglomerates can be found in behavior D. Behavior D occurs when a student cannot suspend the association of a letter with its complex referent, even though it is expedient or essential to do so.

Beth, a calculus student, is working, in the following transcript segment, on the same Books and Records problem that Ann worked on (example 2 in Appendix A). The solution is done out for the student with the "trick" conclusion. She is very troubled by the equation 2B + 6B = 40 which appears in the text of the problem because, as she says, "why would you multiply the cost of each record [$6] times the amount of books [B]." Even though the amount of books equals the amount of records "B is not equal to R because um, you cannot substitute B for R or R for B...and have everything else be right." It appears that she cannot let go of the fact that the referent of B is "books" and thus B should in no way be related to any attribute of records, specifically their 6 dollar cost. B has more meaning than just its quantitative value. Eventually Beth correctly recognizes that 5 books and 5 records were bought. But she is still convinced that the algebraic solution that was given to her is incorrect. She is still troubled by the equation, B = R.

196 S: ...B does not equal R because B is a book and R is a record so a book doesn't equal a record [1].

197 I: Okay. B is a book and R is a record?

198 S: Right. A-an amount...an unspecified amount...it could be zero, it could be 50 million, but in this case, 5 happens to be an answer [2].

199 I: Okay. But just because both of them are 5, they're still not equal?

200 S: Right. Because books aren't equal to...Not if-it's just like saying up..to-you know, tomato is to an orange.

201 I: Uh-huh, uh-huh...uh-huh.

202 S: You know, the amounts can happen to be the same but the just the amount of-of the book [3]...I don't know. I'm getting myself in deeper with this...(9 secs)..the amounts here are the same but they're not the amounts of the same thing [4].

The following numbered comments refer to portions of the above transcript.

1. This is a use of letter consistent with category III.

2. Now the letter is quantitative in value. Note that she says 5 is the answer. She does not identify what it answers; she does not indicate what it is the amount of.

3. In the phrase "the amount of <u>the</u> book," she still seems to be referring to a single book.

4. This last sentence is a key indicator of behavior D.

As a final example of behavior D, consider the following transcript segment. In it, Maria, a statistics student who had had a semester of precalculus, is working on the above Books and Records problem but in a version that does not provide her with the solution (example 5 in Appendix E). She is incapable of solving the problem algebraically at first and resorts to an arithmetic solution. She then finds an algebraic solution by closely modeling her algebraic equations after the arithmetic ones. In this solution, X is the number of records and Y is the number of books. At one point in the solution, she changes the Y to an X. Afterwards, the following dialogue ensued.

090 S: It happened to come out right. The only way I figured it out was because I did it

[the arithmetic solution] by trial and error first.

091 I: Yeah.

092 S: And then I worked backwards. That was the only way.

093 I: Well, how do you feel about this procedure? Do you feel that it is the right procedure? Do you feel pretty confident about that?

094 S: I'm not sure if you could do that, if you could just say, if you could just change Y to X and say X is equal to Y therefore the values are going to be the same.

095 I: Uh-huh. Why might you not be able to do that?

096 S: Because uh-you're dealing with two different things, records and books. And then you put a variable for each one and records aren't books and books aren't records. So er, you know, the variables should be different. But if the values are the same, well that's the question I have. Can you make X equal to Y and switch the Y and make it X so you could combine the two?

Even though the values of the letters are the same, it is unclear to Maria whether they can replace each other because their referents are different. Thus, the referent seems to add a burdensome semantic load to the letter.

<u>Results of the refined analysis</u>.

The transcripts that have been used thus far in the paper are noteworthy in the degree to which they illustrate the various behavioral criteria. They are not, however, exceptional. Each of the above students demonstrated similar conceptions in other problems they attempted. Those students not included among the above examples also exhibited variations on the same kinds of behaviors. In fact, eight of the nine students who were interviewed exhibited at least three of the four behavioral criteria. Thus it could be conservatively stated that eight out of the nine students demonstrated an ill-defined, nebulous view of the letters with which they were working. The letters, for them, were associated with both qualitative and quantitative attributes of the complex referent. Furthermore, there is a sense that these various attributes were not recognized as being distinct from each other but rather as parts of some undifferentiated conglomerate.

	Behaviors			
	A	B	C	D
Beth	x	x		x
Maria	x	x	x	x
Ray	x	x	x	
Janet	x	x	x	x
Paul	x	x		x
Eileen	x	x	x	
Margaret	x	x	x	
Victoria	x	x	x	
Liz	x			x

Table 1: Distribution of observed behaviors.

Table 1 shows the distribution of observed behaviors. Note that, as mentioned above, all nine students exhibited behavior A, a lack of a clear stable, quantitative definition of variables being used. Note also that all students except Liz exhibited behavior B.

Some of the implications of these data will be discussed in the final chapter of this dissertation. It is fair to say at this point, however, that according to the standard set up by this author, the clinical interview data strongly support the hypothesis that many students (in this case, at least eight out of nine college statistics students) view semantically laden letters as undifferentiated conglomerates.

Diagnostic Tests

The purpose of this stage of the study was to see whether additional data from written diagnostic tests would provide more insight into or supporting evidence for some of the discoveries made in the clinical interview phases of the study. The written results from two related diagnostic tests were scored and analyzed. The first test was administered to 101 college students enrolled in the first semester of a calculus course designed for students in the social and biological sciences. The test was given towards the end of the semester. The second test was given to 153 different students enrolled in the same calculus course.

These students were just beginning their semesters.

The diagnostic tests were originally designed with more goals in mind than are relevant to this study. Much of the data obtained, therefore, is only tangentially related to a discussion of undifferentiated conglomerates. These ancillary data will be presented in Appendix H. Those goals that were relevant to the current discussion are the following:

1. To obtain raw data on the overall success rate for the types of problems that were given in the clinical interviews.

2. To see to what extent the four behavioral criteria given on pages 100 and 101 are observable in student responses and to determine whether a correlation exists between the exhibiting of some of those four behaviors and success or failure on the problems.

In each set of tests, each student was given two of four possible problems. The format of the tests is given in Appendices F and G. The problems are copied here for the sake of the discussion.

<u>Problems from the First Diagnostic Test (Appendix F)</u>

1. A Biology teacher bought a collection of frogs and turtles. The number of frogs that he bought was three times the number of turtles that he bought. Frogs cost 3 dollars each and turtles cost 6 dollars each. He spent $60.00 altogether. How many frogs did he buy?

2. A woman had a container of red and green blocks that weighed a total of 91 ounces. Red blocks weigh one ounce each. Green blocks weigh three ounces each. The number of red blocks was four times the number of green blocks. How many red blocks did she have?

3. A Biology teacher bought a collection of frogs and turtles. The price of one frog was three times the price of one turtle. He bought three frogs and six turtles. He spent $60.00 altogether. What is the cost of one frog?

4. A woman had a container of red and green blocks that weighed a total of 91 ounces. She had one red block and three green blocks. The weight of a red block is four times the weight of a green block. How much does one red block weigh?

Problems from the Second Diagnostic Test (Appendix G)

1. A biology teacher bought a collection of frogs and turtles. The number of frogs that he bought was four times the number of turtles that he bought. Frogs cost one dollar each and turtles cost three dollars each. He spent $91.00 altogether. How many frogs did he buy?

2. A woman has a container of red and green blocks that weighed a total of 60 ounces. Red blocks weigh three ounces each. Green blocks weigh six ounces each. The number of red blocks was three times the number of green blocks. How many red blocks did she have?

3. A biology teacher bought a collection of frogs and turtles. The price of one frog was four times the price of one turtle. He bought one frog and three turtles. He spent $91.00 altogether. What is the cost of one frog?

4. A woman had a container of red and green blocks that weighed a total of 60 ounces. She had three red blocks and six green blocks. The weight of a red block is three times the weight of a green block. How much does one red block weigh?

Students were randomly given tests containing either problems 1 and 4 or tests with problems 2 and 3. Each problem was given on a separate sheet of paper and the order of the problems was randomly mixed.

Raw data.

Some of the data that help to put a general perspective on overall student ability on these types of problems are given in Tables 2 and 3.

	No. of problems (2/person)	No. correct	% correct
Test 1	202	102	50%
Test 2	306	144	47%
Total	508	246	48%

Table 2: Overall Success Rates on Two Diagnostic Tests

	No. of people	No. of people getting: both problems correct	one problem correct	no problem correct	score
Test 1	101	34 34%	34 34%	33 33%	1.01
Test 2	153	44 29%	56 37%	53 35%	0.94
Total	254	78 31%	90 35%	86 34%	0.97

Table 3: Individual Performances on Two Diagnostic Tests

Table 2 is the overall success rate on these tests. Table 3 gives data on individual performances. It tallies how many people got both, one, or no problems correct. In the column labeled "score" in Table 3 are the average number of problems done correctly.

These data can be briefly summarized by saying that overall, there was less than a 50% success rate on these algebra problems. Student, on an average, got slightly fewer than one out of two problems correct with 34% of the students getting no problems correct.

Evidence for the four behaviors in the written data.

The second goal of this phase of the research was to answer questions like the following. Are the above results in any way related to student conceptions of semantically laden letters? Is there evidence in the written data for any of the four behavioral criteria indicating the conception of letters as undifferentiated conglomerates?

In attempting to answer these questions, some issues pertaining to research methods became self evident. Difficulties were encountered in attempting to analyze written data in the same manner in which the clinical interview data was analyzed. It was found that written data alone tended to be more ambiguous than clinical interview data vis-a-vis revealing what the student may have been thinking. A digression is in order at this point to discuss some of those issues.

As is seen in interview data, identical written responses can be derived from very different conceptual backgrounds. The following is an example. The sentence, "the number of frogs that he bought was three times the number of turtles that he bought" is correctly translated into the equation F = 3T, where F and T mean the number of frogs and turtles bought, respectively. Can it be assumed, however, that when a student writes F = 3T, he or she is thinking of the letters in that way? The answer is no, not necessarily. One student, on the written diagnostic test, translated that sentence with "Frogs = 3 Turtles." It seems highly likely that other students who write F = 3T may be thinking in a similar way as this student. In fact, when one looks at the results of the clinical interviews, that students regard the F and T as "frogs" and "turtles" rather than "number of frogs" and "number of turtles" is not only possible but even probable.

As was also seen in the clinical interviews, a student could get a problem correct despite the fact that s/he may have been very confused about the meaning and use of the letters. The following is a problem on the diagnostic test and a solution found among the written responses:

> A boy bought a collection of frogs and turtles. The price of a frog was three times the price of a turtle. He bought 3 frogs and 6 turtles. He spent $60 altogether. What was the price of a frog. Solution:

$$F = 3T$$
$$3F + 6T = 60$$
$$9T + 6T = 60$$
$$T = 4$$
$$F = 12$$

This student thus got the correct answer and ostensibly has a good understanding of how letters are used in equations. But it is possible that that is not the case. This is indicated by the following scenario.

In a clinical interview, Ray also did the above problem. He struggled for quite a while with several false starts. He then found the above method and did it quickly and confidently. He was then given an analogous problem. The following discussion ensued:

> 031 S: (reading) A girl had a bag of red and green blocks that weighed a total of 91 oz. She had 1 red block and 3 green blocks. The weight of the red block is 4 times the weight of the green block. How much does 1 red block weigh?--This is similar to the last problem. So, the weight of a red block is 4 times the weight of a green block so, the same thing (writes R = 4G). R equals 4 times G and then you have...She has 1 red block (writes R) plus 3 green blocks (writes + 3G) equals 91 oz. (Has written R + 3G = 91).

Note that when he read 3 green blocks, he wrote 3G. There is no indication that multiplication is going on between 3 and the weight "G." Rather, there is a sense that G is a label for green blocks. However, he does get the problem correct, properly substituting 4G for R and solving for G. The interviewer then asks, referring to the equation R + 3G = 91:

> 034 I: What does R stand for?
>
> 035 S: It stands for a red block.
>
> 036 I: A red block. And G stands for...
>
> 037 S: That's a green block.

If Ray's written work alone were analyzed, it, like the student's work on the written test given previously, would seem quite correct. It is only when Ray's verbal comments are taken into consideration that his misconceptions become apparent. Incidentally, the next problem Ray worked on is the one from which the transcript beginning on page 110 was taken. In that problem, Ray's lack of understanding of letter use gets him into more trouble.

One student, in answering the problem on page 123 on the diagnostic test wrote:

$$3 \text{ frog} + 6 \text{ turtle} = 60$$
$$3 (\quad) + 6 (\quad\quad) = 60$$
$$3x + 6y = 60$$

It seems clear that x does not mean, as it should, the price of a frog. Rather, it means "frog." How does one tell, then, if the student whose work is on page 123 properly understands the meaning and use of the letter F?

Thus, as indicated above, the written work alone is not always a reliable indicator of the student's conceptions. The differences between written and clinical interview data are underscored further when one attempts to use the behavioral criteria that had been developed for the clinical interviews in analyzing written work. Many of these behaviors have a verbal component, making it difficult to observe the behavior in the written work. With the exception of behavior A, the amount of evidence in written work for the other behaviors is greatly deflated from what it had been in the clinical interviews. Behavior B is defined by the eleven observed categories of letter use. Many of those categories, however, are rarely evidenced in written work. For example, in the dialogue with Ray given on page 124, when Ray writes R = 4G, R seems to mean the weight of a red block (category IV). The same assumption could be made about R in R + 3G = 91. However, Ray's verbal addenda is that R is "a red block," which fits category III. Category III is rarely evidenced in written work alone. On the diagnostic tests, the closest evidence to category III was the written phrase "red block = R." This, however, is not conclusive evidence of category III. It can be argued that "red block" is just time saving shorthand for what the problem solver really meant, e.g. "the weight of a red block."

The same is true for many of the other categorizations. Thus one is less apt to detect the shifts in letter use inherent in behavior B.

Behavior C requires that 2 quantitative uses for the letter be juxtaposed without the students' recognition of any contradiction. This does seem to occur often in the written work. However, the students' thoughts and views are left for conjecture. One student wrote the following series of equations:

$$r + g = 91$$

$$1r + 3g = 91$$

$$1r + 4(3g) = 91$$

Does each successive equation imply a refinement and rejection of the one before or is there a belief that all three equations are correct but only the last one is needed to solve the problem? Without the verbal input, this remains unclear.

Behavior D is even more closely linked to verbal data. It is evidenced when a student verbally reports something akin to "I cannot replace B with R because even though the numbers are the same, books are different from records." In written work, it is often the case that students do not appropriately replace one variable with another. However, without the verbal input, it cannot be claimed that behavior D is occurring.

Nevertheless, even though the extrapolation of the categories and behavioral criteria to written work is problematic, some significant results can still be found. One such result is a very simple one but one which has strong pedagogical implications. It is the fact that of the 245 students who used algebra in at least one of the problems they attempted, 212 of them (87%) exhibited behavior A in at least one problem solution. That is, 212 students either gave no definition for the variables they used or the definitions given were not quantitative. It is interesting to note that those students who consistently gave clear, quantitative definitions for the letters they used scored significantly better on these problems than did students who exhibited behavior A. The former students had a mean score of 1.54 whereas the latter students had a mean score of 0.88. (The reader will recall that a student was given a score of 2 if both problems were done correctly, 1 if one problem was correct, and zero if no problems were correct.) A two-tailed t-test establishes that the difference was significant at the $p < .001$ level.

In addition to looking for behavior A in the written responses, the written data were scrutinized for evidence of the other behaviors as well. It was found that 125 of the 254 students (49%) exhibited at least 2 of the 4 behaviors on at least one of their two problems. The mean score for these students was 0.70 whereas tha mean score for the remaining students was 1.22, a difference which was significant at $P < .001$.

The reader should be reminded, however, that because the three behaviors B, C, and D are evidenced much less frequently in written work than in clinical interviews, these latter statistics are probably very different than they would be had the data been collected via clinical interviews.

One other finding in the written data should be noted: that it is clear that students' conception of semantically laden letters goes beyond a "labels approach." They do evidence the use of letters as labels quite often. But they also treat the letters quantitatively and demonstrate other qualitative applications for which a "labels" description is inadequate.

Because of the inadequacy of the written data, one cannot assert with the same degree of confidence as was the case with the clinical interview data, that students have improper conceptions of semantically laden letters. Specifically, it cannot be established from the written data alone whether the conception of semantically laden letters as undifferentiated conglomerates is prevalent among non physical science undergraduates. However, there is nothing in the data to contradict this. Certainly, the poor showing in general and the even poorer algebraic showing in particular could be predicted by a hypothesis that says that students conceive of letters as having amorphous and undifferentiated meaning. These students' overwhelming tendency to neglect to define the letters they are using as values of quantitative attributes of the referents could also have been predicted by that hypothesis. That the written data does not contradict, but, if anything, lends support to the clinical interview data gives this latter data added importance.

CHAPTER VI

IMPLICATIONS AND DIRECTIONS FOR FUTURE RESEARCH

As is often the case with studies that are exploratory in nature, the above research raises more questions than it answers. This chapter will attempt to outline what some of those questions are and to suggest possible directions that future research could follow.

This chapter is organized so as to parallel the development of the thesis thus far. It is broken down into four sections. The first section pertains to questions arising out of the discussion of "matter meant" in Chapter II. The second section deals with some of the implications of the textbook review in Chapter III. In the third section, some implications of the research reported in Chapters IV and V are discussed, specifically focusing on those pertaining to the students' views of semantically laden letters as undifferentiated conglomerates. A fourth section has been added to discuss the concept of an "undifferentiated conglomerate" in a more general context and to raise questions as to its significance for cognitive psychology.

Matter Meant: Implications and Directions for the Future.

In the first part of Chapter II, the works of Küchemann and Tonnessen were discussed and critiqued. A conclusion of that analysis was that in assessing the meaning of an algebraic letter, the context in which that letter is found must be taken into consideration. Furthermore, the importance of assessing what is meant by a letter independently from what students perceive the letters to mean was emphasized. This led to the development of a list of five analytical questions which is recapped here for reference.

1. What is the abstract domain of the letters?
2. What is the dimension of the problem?
3. How is the letter perceived to vary?
4. What is the solution set for the letters?
5. Is there semantic meaning attached to the letter that is different from its meaning in the abstract? (i.e., is it semantically laden?)

An assessment tool like these five questions would be invaluable if it was found to be reliable in predicting differences in the meanings of two different letters in two different problem contexts. In Chapter II, it was suggested that any difference between two algebra problems vis-a-vis their use of letters might be explained by variations in the answers to one or more of these five questions. This dissertation did not test the veracity of that claim. An attempt at doing so might prove to be an interesting and relevant basis for future research.

In addition, one might ask whether a more efficient assessment tool could be developed. That is, is there a simpler and quicker means of determining what is meant by a letter that would still be able to uncover all differences in use and meaning between letters in different contexts?

Even if a simpler assessment tool can be found, it seems clear that it will remain relatively complex because the uses of letters in precalculus algebra are extremely disparate and themselves complex. This complexity alone suggests one of the most important questions to be asked by this dissertation. If the meaning and use of letters is so complex, why, as is shown in Chapter III, is so little curricular time spent in presenting the concepts in the schools? This question becomes even more compelling when the data in Chapters IV and V that suggest the extent of students' confusion about the meaning of letters is taken into consideration. More will be said about this in the next sections of this chapter.

Before doing so, some of the open research questions implicit in the information presented in Chapter II will be discussed. As mentioned at the beginning of Chapter IV, each of the five questions that comprise the assessment tool themselves suggest a myriad of research questions pertaining to student conceptions. Chapters IV and V attempt to answer just one such question. (What are some student conceptions of semantically laden letters?)

The following are just some of the additional research questions that could be asked:

1. If, as the first of the five assessment questions

implies, letters can have different types of abstract domains, one might ask what effect does changing the domain of a letter have on a student's ability to solve problems? Are students, in general, capable of perceiving the difference between integral, rational, or real domains? Do students understand what is meant by a domain and do they find it counter-intuitive that a letter can be replaced by more than one value?

2. Similarly, as implied by assessment question two, one might ask what effect does the dimension of a problem have on students' understanding and their ability to solve the problem. Not surprisingly, data that has already been collected (Clement, 1981 and Küchemann, 1978, p.24) suggests that students do worse on problems involving more than one variable. The results from the diagnostic test reported in the previous chapter clearly indicate a preference for and a greater adeptness at solving word problems using one variable rather than two. Since inherent in most functions are relationships in at least two variables, and since functions play such a central role in calculus and higher mathematics, students' apparent discomfort with expressions having more than one variable gives cause for concern and thus grounds for further research.

3. Also central to calculus and much of the higher mathematics is the concept of continuity alluded to in the

third assessment question. In taking a limit of a function, one must not only allow for a letter to have a continuous domain but must also allow that letter to vary continuously over that domain. Kaput (1981) has begun to look at the issues of student conceptions of functions and continuity vis-a-vis their understanding of algebraic letters. Confrey (1980, p.228) has conducted clinical interviews examining first-year college students' concepts of number by asking them to solve problems relating discrete and continuous. Kerslake (1977,pp.22-25) has investigated the extent to which students view graphs as having an infinite set of points. All of these studies are at least tangential to the question of whether students are capable of perceiving letters to vary, and if so, if students are capable of perceiving letters to vary continuously.

4. The fourth assessment question, that pertaining to the solution set for a letter, raises similar kinds of questions. As mentioned earlier, Collis (1978, pp.223-24) has noted that many students have an inability or reluctance to "accept a lack of closure." These students have difficulties with problems whose solutions are expressions rather than numeric answers. Küchemann (1978, p.25) has shown that letters that are left unsolved are often ignored by students. As was shown in Chapter III, some texts define variables in terms of their solution

sets. That is, a letter is defined as the number or numbers it ultimately will be found equal to. This over-association of a letter to a numeric answer might foster the difficulties mentioned above. On pages 42 and 43 of this dissertation, the many different types of solution sets found in algebra were presented. It is possible that the extent to which a student is aware of and open to the different possibilities for solution sets could be reflected in their ability to deal with functions, unsolvable expressions, and, in general, their ability to accept a lack of closure. This area could be a basis for future investigations.

5. Finally, we come to the fifth question and possibly the richest one in terms of a potential for future research questions. Semantically laden letters occur in those problems where the algebra is modelling some world scenario. The variations of those world scenarios being unlimited implies unlimited variations in meanings for semantically laden letters. Thus, it might in fact be the case that the fifth question should be reworded to read, "Is there semantic meaning attached to a letter, <u>and if so, which type of semantic meaning</u>?" Those problems used in the research reported in Chapters IV and V make up only a small subset of all possible algebra word problems. Thus a goal for future research might be to test what effect changing the type of semantic content has on students' conceptions

of the letters and their ability to solve problems. Ultimately, a taxonomy or classification scheme for the types of semantic load a letter can carry could be created. For example, the issue was raised in Chapter II (page 46) as to whether the "distance" between the semantic and abstract referents of a letter had an effect on how that letter and the problem in which it was found were understood. The possibility of defining a metric to measure that "distance" was suggested. This is just one way in which the type of semantic meaning attached to a letter can vary. Whether the letter refers to a discrete or continuous entity in the world, whether the letter refers to an intrinsic or an extrinsic measurement, and whether the letter refers to a familiar or unfamiliar entity, could all be factors that are reflected in the meaning of a semantically laden letter.

Whether the findings from Chapters IV and V would be significantly altered if the semantic content of the problems were altered along some of the above dimensions is a vital and important question and suggests a logical next step in the investigation of student conceptions of semantically laden letters. More will be said on this issue in the third section of this Chapter.

In summary, in Chapter II of this dissertation is an assessment tool that is purported to be one that can determine efficiently and thoroughly the meaning of an algebraic letter vis-a-vis the problem context. The extent to which this is true should be tested in a future study. This assessment tool is made up of five questions. Each of these questions themselves suggest several additional research questions vis-a-vis students' understanding of algebraic letters.

Matter Taught: Implications of the Textbook Review.

The information presented in the textbook review in Chapter III uncovers a fascinating, albeit disturbing, pedagogical paradox. One conclusion that can be drawn from the information presented in the chapter is that there are literally dozens of different approaches to introducing and emphasizing letters in precalculus algebra. Very seldom do two texts approach the topic in the same manner. This points again to what had been suggested by Chapter II; that the topic of letters in precalculus algebra is impressively complex.

A second conclusion that was drawn in Chapter III was the fact that a large majority of the texts spent very little space and time defining and developing the concept of a variable. The pedagogical paradox is this: If the concept of variable is so complex, why is it presented so offhandedly and simply. If, for example, there are several different types of domains that letters can have throughout the algebra curriculum, why is it often the case that only one is presented? If, in application, letters can be constant, can vary discretely, or can vary continuously, why is so little attention paid in the texts to those differences? If so many types of solution sets are possible, why is it often the case that those distinctions are assumed to be tacitly understood?

One answer to these questions might be that high school students are not intellectually prepared to deal with the concept of variable in all of its complexity. Thus, no emphasis is placed, for example, on the concept of continuous variability because students are not ready to deal with the distinctions between discreteness and continuousness. However, even if that were true, the texts, it seems, should consciously and explicitly try to build those bridges that would help carry students towards the more sophisticated understanding of the concept of variable. An implication of the information presented in Chapter III is that this is, for the most part, lacking. That is, most textbooks are guilty of an "error of omission" by spending so little explicit space and time developing the concept of variable.

Of equal interest and significance, but more difficult to pin down, are those "errors of commission" found in the textbooks. "Errors of commission" here refers to those things found in the textbooks that may inadvertently foster erroneous conceptions in the students vis-a-vis their understanding of the uses of letters in algebra. Several examples were given in Chapter III. For instance, it is possible that teaching that a letter can be replaced by more than one number may be enhanced by the cannon example on page 59. However, it is possible that since the cannon can only "shoot" one discrete number at a time, and since no other illustrations are given in the text that imply a continuous replacement of the letter, the example of the cannon might be counterproductive in teaching that letters can vary continuously. In that sense, using the cannon may be an error of commission as defined above.

In the first section of this chapter, many research questions were raised pertaining to student conceptions of variables. Each of those research questions could have a corollary question that asks, what, if anything, do the textbooks do, either by omission or commission, to foster particular student conceptions. Those corollary questions in turn each suggest other questions pertaining to curriculum development. The reader will agree that the topic of letters in algebra quickly becomes a Pandora's box, the number of research questions stemming from it growing exponentially. For that reason, the rest of this section will focus on only some of those questions, especially those pertaining to the textbook review that are related to the findings presented in Chapters IV and V. The following is a list of and description of those questions.

1. Is there a relationship between the type of textbook used and student understanding of semantically laden letters? This question, in a general way, asks if there is an association between particular texts and student understanding. Because of the enormous quantity of textbooks, this question would be unwieldy to pursue. A compromise might be to compare some prototypical but disparate texts (e.g. Foerster vs. Beberman vs. Dolciani, etc.)

2. What specific curricular content and strategies for word problems can be associated with students developing a view of letters as labels or letters as undifferentiated conglomerates? In Chapter III, it was suggested that introducing word problems with pure number problems, wording problems so as to avoid difficulties in translations, using a key word matching strategy for translations, and simply giving the student very little experience with problems requiring the use of semantically laden letters may all be associated, either individually or collectively, with student misconceptions. Determining the extent to which this is true could be the basis of a rich, albeit lengthy research project.

3. What applications for letters in precalculus math texts can be associated with students developing a view of letters as labels or letters as undifferentiated

conglomerates? Kaput (1982) has noted that letters in Geometry sometimes refer to variable entities (e.g. an unknown angle) and sometimes are the name of static entities (like a point or a specific angle). He suggests that the lack of distinctions between various disparate uses of letters in Geometry and all of high school math might very well be associated with the types of student confusion described in this thesis.

Similarly, an approach to letters in algebra consistent with what Galvin and Bell refer to as fruit salad algebra (described on page 11 of this thesis) might also be related, possibly causally, to a view of letters as labels and/or as undifferentiated conglomerates. As mentioned in Chapter II, Küchemann (1981) has found several blatant examples of texts that teach "fruit salad algebra." But even if a text does not explicitly employ "fruit salad algebra, it nevertheless is likely that some math teachers read an equation like 2x + 6x = 8x as "two x's plus six x's equals eight x's," allowing their students to infer that letters refer to concrete things.

Related to the use of letters in "fruit salad algebra" is the more mathematically legitimate use of letters as standing for a unit of measure. Gillman (1981) in response to an article by this author (Rosnick, 1981) wrote;

> I think I now know the main source of the confusion [that causes students to write the reversed equation 6S = P for the Students and Professors problem]: surely it must be the students' long familiarity with 'dimensional'

equations such as 1ft = 12in, 1 m = 100 cm, and so on.

This is not an uncommon response from math teachers who have seen the data. However, in over 50 clinical interviews on Students and Professors type problems, there was evidence in only one interview that students who write 6S = P are modeling their solutions on 'dimensional' equations like 1ft = 12in. Therefore, it seems unlikely that familiarity with dimensional equations is a major source of error in Students and Professors type problems.

Nevertheless, equations like 1ft = 12in do have some noteworthy similarities with students' use of letters as "conglomerates" in that several different meanings can legitimately be attached to the symbol "ft." For example, if the equation is seen as shorthand for the sentence "one foot equals twelve inches" then "ft" is merely an abbreviation for the word "foot." If 1ft = 12in is seen as what Gillman calls a "dimensional equation" then "ft" can be thought of as a standardized unit of measure. As Clement (1979, pp.4-5) has noted, 1ft and 12in are equal in the sense that they have the same absolute lengths.The notation, "ft" can be defined as the absolute measure of one foot, measured for example, in centimeters. Thus 1ft = 12in is read "one times the absolute length of a foot is equal to 12 times the absolute length of an inch" or alternatively "one times the number of centimeters in one foot is equal to 12 times the number of centimeters in one inch". Finally, "ft" can refer to a physical manifestation of a foot (as in a ruler). Thus one foot equals twelve inches(1ft = 12in) in the sense that a one foot ruler can be the exact same thing as a ruler with twelve juxtaposed inches. Thus the symbol "ft" can have an array of legitimate referents.

To what extent the use of dimensional equations, the teaching of "fruit salad algebra" and the multiple uses of letters in geometry can be associated with a students' view of semantically laden letters as undifferentiated conglomerates is an open question but a crucial one for ascertaining some of the pedagogical causes for the view.

4. Both questions 2 and 3 above are "negative" questions in the sense that they ask what curricular content and aspects of the texts are associated with student misconceptions. It is just as crucial, however, to ask the "positive" question, what in the texts can be associated with a more "correct" understanding of how letters are used? Does, for example, an approach like that of the SMSG text given on page 74 of this thesis, that attempts to get the student to focus on the specific quantitative meaning of a letter, achieve better results?

In summary, it seems clear that many research questions can be raised that could address the issue of which of the many curricular styles of presenting letters described in Chapter III can be associated with either correct or incorrect student conceptions. In addition, a more general pedagogical question should be asked: if it is true that the uses for letters in algebra are so varied; if it is true that the use of letters symbolically is crucial to mathematics as a whole; and if it is true that the students are as confused as is strongly suggested by the data in Chapters IV and V, why is so little textbook space devoted to the presentation of the concept of variables?

Matter Learned: Implications of the Findings in Chapters IV and V.

In this, the third section of this chapter, the implications of the findings reported on in Chapters IV and V will be discussed from two perspectives. First, some specific suggestions for the future amplification and development of those findings will be given. Second, some pedagogical issues pertaining to mathematics education in general will be raised.

Future amplification and development.

The findings reported on in the previous two chapters are limited in that the class of problems that were used as a basis for the research is a relatively narrow one. As has been suggested earlier, there is much to be learned from testing these findings in wider domains. One might ask, for example, how helpful the list of eleven categories of letter use would be if the type of algebra word problems were significantly changed (e.g., "time on the job" problems or "age" problems). In what ways would the list be expanded? Are there some of the eleven categories of student behavior that are not evidenced?

Similarly, one can ask whether there would continue to be evidence for any of the four behavioral criteria pointing to the student conception that letters refer to undifferentiated conglomerates and whether some other typical errors found in algebra can be attributed to this misconception.

More generally, how a view of semantically laden letters as undifferentiated conglomerates affects students' understanding of variables when found in Geometry, Statistics, or Calculus are open questions that could be pursued. It is very possible, for example, that part of the difficulty in teaching the concept of a random variable in Statistics and Probability is connected with students' unawareness of the need to carefully define one's variables.

As another example, consider the "maximum/minimum" problems taught in most calculus courses. Most teachers of calculus would agree that these problems are among the most difficult to teach. Certainly, a major part of the problem is that students do not sufficiently understand the concept of a derivative. Misconceptions concerning functions and the continuous nature of the variables are also likely contributors to confusion. In addition, however, is the fact that these problems require the use of semantically laden letters. In many problems, like the proverbial "fence" problems (what is the largest area given a fixed perimeter or what is the smallest perimeter, thus cost of materials, given a fixed area), there is more than one relevant attribute to consider (perimeter versus area, or quantity versus cost, etc.). It is reasonable to expect that the students would be at least as confused as they seem to be on the algebra problems presented in this dissertation.

In fact, in any mathematical domain where letters are used, especially where they are semantically laden, it is reasonable to expect, based on the data presented in chapters IV and V, that students will view them as undifferentiated conglomerates and that cognitive difficulties that arise in that domain might be related to that view. These are all hypotheses that bear testing with further research, the results of which would be very useful to curriculum developers.

In addition, one must wonder what implications these data have for other subjects like Chemistry, Physics and the like. In Chemistry, for instance, letters are often used to stand for things. The symbol, H20 signifies a molecule with two atoms of hydrogen and one atom of oxygen.

However, there are certainly plenty of places in chemistry where letters take on more traditional roles as variables (in an algebraic sense). One might ask whether the legitimacy of using letters to stand for things adds to students' confusion concerning the use of letters as quantitative variables.

Most of the research studies that have been suggested thus far in this section are diagnostic in nature in that their purpose would be to better understand student behavior and develop theories about student conceptions. Another whole area of research is possible that would be more prescriptive in nature. The question central to such research would be what curricular strategies work best in developing in students a proper conception of letter use in algebra. For example, one might, as has been suggested in the previous section, compare current texts to find which are more successful in building proper conceptions. On the other hand, one might decide that whole new devices need to be developed for that purpose.

One area that, according to some, holds promise for new ways of developing the concepts of variable and function in students is computer science. Clement, Lochhead and Soloway (1980) have found that students who err on translation problems like the Students and Professors problem often do significantly better when asked to write computer programs that use the same information.

(pp.9-11) One hypothesis that they give for the higher success rate with computers is that the computer languages require a more explicit definition of the symbols that are used.

Pappert in Mindstorms says that computers are exceptional devices with which to present the concept of variable. It is the recursiveness inherent in many computer programs that bring out the essence of the variability of a variable. (pp.69-75) (Pappert does not address the issue that replacements for the variable are done discretely rather than continuously by the computer.)

An important question that remains to be pursued is whether there is any carry over from what students learn in the computer languages to their facility with mathematics. It must be asked whether a better understanding of variables when found in the context of a computer program translates to a better understanding of variables in general. Furthermore, it must be asked whether learning about variables through computer programming helps deal with the issue of student difficulties with semantically laden letters.

General pedagogical implications.

This section thus far has dealt with specific implications of the findings reported on in Chapters IV and V, primarily in terms of directions that future research could take. The remainder of this section describes more general implications which argue for the refocusing of the entire mathematics curriculum. Three arguments will be made. They are as follows:

> 1. A goal of math education should be that students ultimately understand the material well enough to be able to apply it to real world situations.
>
> 2. A traditional curriculum that focuses on the development of basic skills has been justified, in part, with an argument that says that understanding evolves from competency in the basic skills; a theory of mathematical osmosis.
>
> 3. The findings reported on in Chapters IV and V seem to refute the theory of mathematical osmosis and thus suggests a reorienting of the basic skills curriculum.

1. Mathematics, when devoid of semantic attachments, can be seen by students as a lifeless, useless mental exercise. It is when mathematics is used to model real life or practical situations that its true power becomes evident. In colleges and universities across the country, more and more students are being required to take more and more mathematics. Ostensibly, academic departments require their students to take courses like Calculus and Statistics because at least some of the course content is applicable to their fields. Therefore, the extent to which students understand and are able to apply concepts and skills they have learned in their math courses must be an indicator of the value of the courses.

2. However, as indicated in Chapter III, the mathematics curriculum today, for the most part is focused

on the "basic skills" of computation and manipulation of expressions. This is justified in part by an argument which says that understanding evolves (as if by osmosis) from competency in basic skills. Saxon (1981) says,

> teaching [an algebraic] skill requires patience and the realization that understanding of concepts is not a prerequisite of the initial development of skills: understanding often follows the ability to do rather than precedes it. (p.1204)

This is the rationale Saxon uses for a text that he wrote that focuses primarily on the development of "basic skills." Saxon predicts that understanding and, presumably, the ability to apply basic skills, would follow from a competency in those basic skills.

Similarly, many college calculus texts, especially those that are designed for students majoring in business or the social sciences, place a greater emphasis on teaching students how to take derivatives than on what derivatives are and how they are used. A theory of mathematical osmosis might say that if students become adept enough at taking derivatives, then their understanding of what a derivative is will automatically evolve.

3. The theory of mathematical osmosis is being challenged by research. Clement, Lochhead, and Monk (1981) for example have collected data that show that many students who fare poorly at problems involving translations from an English sentence to an algebraic expression or vice

versa can, nevertheless, quickly and accurately take the derivative of a fairly complicated function. That is to say, many students do well on problems involving symbol manipulation but much less well on semantically laden problems.

Similarly, the data presented in chapters IV and V seem to refute Saxon's rationale for a focus on basic skills. Relatively speaking, student basic skill levels were adequate or better. Yet their understanding of the letters they use seems to have been greatly lacking. There seems not to have been an evolution from a facility with "basic skills" to "understanding" and the ability to apply those basic skills.

Most of the students who were interviewed showed little difficulty in manipulating and solving equations and performing other skills of basic algebra. Where their difficulties became evident was in the creation and or interpretation of those equations. It is thus at the interface between the semantic content of a problem and its algebraic representation where students' skills appear to be most lacking. Yet it is the ability to interface semantic content with an algebraic representation that is crucial to the application of mathematics to practical situations.

If one accepts that the ability to apply mathematical skills is itself an essential skill, then the data reported in chapters IV and V are extremely important. They indicate that the lack of clarity and stability in the definition and use of letters to represent semantic referents is likely to be a significant factor in students' difficulties with word problems. The association between some of the behavioral indicators of a students view of letters as undifferentiated conglomerates and poorer scores on the written diagnostic test is one of the indicators of that. If one views an inability to solve word problems as symbolic of a general inability to apply the basic skills, then students' misuse of semantically laden letters is itself symbolic of a significant shortcoming of the mathematics curriculum. The very premise for teaching these students mathematics, i.e., that they will be able to apply some of what they learn, is called to question.

It is this that suggests the need for the reorienting of the mathematics curriculum away from a focus on basic skills and towards a goal of "understanding" and the ability to apply mathematical tools to real world situations. Curriculum developers must question, for example, the value of requiring business and social science majors to learn calculus when the data show that these students' understanding of algebra as it models the world is greatly lacking. Furthermore, general problem solving skills must be emphasized at all levels, especially those skills that are required for the solution of semantically laden problems.

Implications for cognitive science.

In the last section, pedagological implications of the data presented in chapters IV and V were discussed. Implicit in that discussion was that students' difficulties may have been caused by shortcomings of the mathematics curriculum. But what about cognitive factors? Are students' difficulties with clearly and stably defining quantitative variables a result of poor learning or is there a deeper cognitive basis for the lack of differentiation?

To answer that question would be beyond the scope of this dissertation. This section is only meant to be grist for the cognitive science mill and will discuss some related issues and research.

Vygotsky, in Thought and Language discusses the topic of concept formation in children. His theories are based in part on the following experiment. Subjects are given blocks of different size, shape, color, and height and each block has one of four nonsense words written on its underside. The words have meaning. (For example, "lag" means a tall large figure.) The experimenter shows one block to the subject, reads its name, and asks the subject to pick out all of the other blocks which he thinks might be called by the same name. After the subject attempts this, the experimenter shows the names on the blocks that were picked incorrectly. The process is continued until the subject learns the "concept" that defines the word.

Vygotsky details three "phases" in the ability to develop true concepts, the most primitive of which is what he calls "heaps". A heap, according to Vygotsky, is an "unorganized congeries" where

> word meaning denotes nothing more to the child than a vague syncretic conglomeration of individual objects that have some how or other coalesced into an image in his mind. Because of its syncretic origin, that image is highly unstable. (pp.59-60)

As mentioned above, this is a very primitive phase in the evolution of a childs conceptual thinking. Surely, most college students would perform Vygotsky's experiment in a much more sophisticated manner than children who create heaps. No claim will be made, therefore that college students think at that level.

But it is nevertheless remarkable how closely Vygotsky's definition of a heap matches the description of an undifferentiated conglomerate. An undifferentiated conglomerate is an unorganized, vague, unstable congeries of various attributes of a complex referent. Variables in mathematics should be defined in an organized fashion carrying very specific meaning, and not refer to "unorganized congeries." Their meaning should be clear and stable, not vague and unstable. In comparison to that of a mathematician, students' definition and use of semantically laden letters seems primitive. It is not unreasonable to suspect then that their general ability to form concepts and their general ability to abstract is also primitive.

One plausible hypothesis that would explain this primitve view, is that these students have not, in some ways, reached Piaget's formal operational level. As mentioned in Chapter I, Gray, Karplus, Renner, Lawson and others have made such an argument. Karplus, for example, based his conclusion ,in part, on poor performances of college students on tasks involving proportional reasoning. These authors might suggest that students' difficulties with semantically laden letters is connected to thinking in an overly concrete way. That is, their over-association of a letter with the concrete referent might be indicative of an overall cognitive difficulty with formal operations.

It should be emphasized that a general indictment of these students' levels of thinking may not be necessary. It might be the case that, though these students have not constructed an adequate conceptual mathematical background, they are still perfectly capable of sophisticated conceptual thought in other areas of their lives. If this proved to be so, it argues against an overall stage or phase discription of intellectual development and, instead, for a cognitive science that in some way compartmentalizes the development of conceptual understanding.

Bamberger and Schön (1978) describe a process in the aquisition of new cognitive skills that they call the "figural-formal transaction." Their study is based on observations and interviews with two subjects whose task it was to first learn how to play "Twinkle Twinkle Little Star" on bells of different pitch and then to develop a symbol system that encodes how that is done. The subjects are an eight year old boy (Jeffrey) and a clinical psychologist (Steve).

The Bamberger and Schön article is relevant to the present discussion in at least two ways. First, it is noteworthy that during the middle stages of Jeffrey's learning, he revealed a primitive thinking not at all unlike that of the college students' "undifferentiated conglomerates." Specifically, Jeffrey invented a notation for encoding "Twinkle Twinkle" but was not careful in differentiating whether a particular symbol stood for a bell, a beat, or a note. That is, he at first did not differentiate whether the notation represented the position of the concrete bell, or represented one hit on the bell, or symbolized the more abstract idea of a musical note. While trying to encode the song, Jeffrey often unconsciously shifted his attention and thus shifted the meaning of the notation from one thing to another, just as college students shifted the meaning of sematically laden letters.

Second, Steve, who had had no formal previous musical training, dealt at first with the problem in a manner not unlike Jeffrey. That is, though clearly capable of sophisticated thought (inferred from the fact that he is a psychologist), his thinking about musical notation began at quite primitive conceptual levels.

Bamberger and Schön believe that what they discovered about the learning of musical notation can be extrapolated to "every domain of human experience where the interaction of apprehended sensory figures and symbol systems is possible --for example, mathematics..." They go on to say that,

> These figures, are situationally dependent, <u>unstable over time and context</u> and across persons, and they are <u>vague</u> in a way that may subsequently be described as a fusion of fixed properties... and of situational function... In order for figures to enter into a symbol system, work must be done... <u>situationally dependent figures must be made into named, differentiated elements and relations which can be held constant over time and context and person.</u> (emphasis added) (p.40)

The "work" that they refer to involves the "figural-formal transaction" that takes the learner from an overemphasis on vague figural representations to a more clearly differentiated symbol system. If one were to apply that theory to the current study, it would seem that college students who view semantically laden letters as undifferentiated conglomerates have not done that work. They are stuck in a primitive form of thinking about algebra even though they may be capable of sophisticated thought elsewhere.

Boas (1911) said in reference to the lack of generalized forms of expression in American Native languages that

> the fact that generalized forms of expression are not used does not prove inability to form them, but it merely proves that the mode of life of the people is such that they are not required. (p.152)

Similarly, the fact that college students have difficulty with the use of algebraic symbols does not prove an inability to do so but may suggest a lack of experience with such concepts. Instead of a general cognitive malaise, students' difficulties might be limited to a family of conceptual tasks. Just how broad is that family that is related to tasks of symbolizing semantically laden entities in algebra is an open question. In the last section, examples were given from Statistics, Calculus, and Chemistry where the symbols being used were subject to interpretations as undifferentiated conglomerates. Bamberger and Schon gave another example in the task of symbolizing music.

Still another example is embodied in certain relations in Physics. Reif (1977) describes four abilities involved in understanding a relation, one of which entails,

> making appropriate discriminations between symbols (either quantities or entire relations) and their referents...so that a particular symbol is correctly assigned to a particular referent, without being either misassigned to inappropriate referents or confused with other symbols. (p.5)

Reif has discovered that many students do not have the above ability to discriminate. Furthermore, he has noted (Reif, 1982) the difficulties that students have in differentiating between different forms of energy, i.e. kinetic, potential, human (as in vitality), etc. Apparently, the term energy is used in an overly generalized fashion whereby students do not discriminate as to what type of energy they are talking about.

This last example is particularly noteworthy in that it goes beyond the problem of symbolization. It may be the case that the concept of an undifferentiated conglomerate would be helpful in explaining many of the instances where people develop vague and amorphous concepts.

Thus, learning more about undifferentiated conglomerates would help in the goal of learning more about the way people think even outside the domains of mathematics and the symbolization process. To understand more about undifferentiated conglomerates would be no small task, but it promises to be of no small consequence either.

C A P T E R V I I

SUMMARY

This chapter will summarize, chapter by chapter, the findings of this dissertation. Before doing so, however, a global statement follows that summarizes and encapsulates the purpose and the tone of the preceeding work.

The introduction to this dissertation alludes to the burdgeoning mathematics enrollment across the country and the fact that the number of major courses of study and career choices available to students who avoid mathematics at all costs is rapidly decreasing. Mathematics courses are less and less the sanctum of what Menger referred to as the mathematically virtuous.

It has been said in the past that the mathematics curriculum has acted as a filter, screening out a select few to work in the technological and scientific fields. Sells (1980, pp.340-41), for example, reported on how that screening process is related to the underrepresentation of women in those fields. But such a filtering process is undesireable, both for political and pragmatic reasons. With more and more people being integrated into technological fields, and with technology and science being integrated more and more into all other fields, the mathematics curriculum must continually be reviewed and strengthened to insure that the imparting of skills of quantitative and logical reasoning be widespread and nondiscriminating.

This dissertation is part of that review process. Mathematical concepts like the use of letters in precalculus algebra that are far more complex and potentially confusing than might at first be anticipated, are the secret scourge of many students. To review concepts like the use of letters in precalculus algebra, not only from the perspectives of the mathematician and curriculum developer but also from the perspective of the student and the cognitive psychologist will serve to aide the future development of, if you will, a mathematics curriculum for the masses.

It is true that a look at the use of letters in precalculus algebra is but a small step towards that global goal. Nevertheless, this dissertation can serve as a model for the review of many different mathematical concepts. To that end, as the dissertation is summarized chapter by chapter, some findings and research principles will be included that can be extrapolated to the researching of other mathematical concepts. Such generalizable findings and principles are underlined throughout the remainder of this chapter.

Central to the organization of this dissertation is the distinction made by Bauersfeld (and described in the introduction) between matter meant, matter taught, and matter learned. To fully understand a concept like that of the use of letters in precalculus algebra, <u>one should investigate independently what is meant by the concept, how the concept is taught, and what students have learned about the concept.</u> To that end, Chapter II is a discussion on what letters mean, Chapter III is a review of how texts teach the concept of algebraic letters, and Chapters IV and V report investigations into how students understand the use of algebraic letters.

Chapter I is a review of the related literature, most of which concerns students' understanding (and lack thereof) of the concept of letter use in algebra. In many of these studies, the translation process inherent in the solution of word problems is described as problematic for students, and students' misconceptions concerning the use of algebraic letters is seen as a significant contributing factor to their difficulty with those problems. Also common to many of the studies is the discovery that students often inappropriately associate algebraic letters with a nonquantitative label for an object. An underlying theme of all the studies reviewed in Chapter I seems to be that <u>it is important to understand student conceptions and misconceptions concerning mathematical topics.</u> That theme also underlies this dissertation.

In Chapter II, the works of two of the authors referred to in Chapter I are discussed and scrutinized.

Both authors, Tonnessen and Küchemann, have worked extensively to describe what is meant by an algebraic letter.

Tonnessen does so with what is described as a general definitive approach. He pares away what he refers to as irrelevant attributes of letters, and develops a formal definition of the concept of variable with mathematical crypticness and clarity. In contrast, this author argues that such a mathematically rigorous but semantically barren definition is not appropriate for understanding what is meant by algebraic letters. <u>One cannot hope to fully understand what is meant by a concept if one pares away the contextual environment in which the concept is found.</u> It is further argued that it is just those "irrelevant attributes" that add to the richness of meaning of the concept of variable.

Küchemann's approach to describing the meaning of a letter is referred to as a specific categorical approach. Unlike Tonnessen, Küchemann creates a somewhat hierarchical taxonomy of letter use in algebra that reflects changes in the context of the problem in which the letters are found. He does not, however, fully acknowledge the role that that context plays in imparting meaning to the letter. Furthermore, <u>he does not fully appreciate the need to distinguish between what a concept truly means and what meaning students give to the concept.</u> That is, his hierarchy mixes matter meant with matter learned.

In quest for a more adequate tool for assessing what is meant by a letter, Chapter II describes five questions that should be asked in determining the meaning of a letter in algebra. They are:

1. What is the domain of the letter?
2. What is the dimension of the problem?
3. How is the letter perceived to vary?
4. What is the solution set for the letter?
5. Is there some semantic meaning that is different from the abstract meaning of the letter? (Letters with added semantic meaning are defined as semantically laden letters.)

These five questions not only serve as an assessment tool but are also used as an organizational device for reviewing texts in terms of how they present the concept of letter use in algebra. In Chapter III, forty-one texts are reviewed including Junior High, Algebra I and II, and college precalculus texts. Each text is analyzed for how each of the five areas are presented. For example, how the texts treat the topic of domain and how often and in what ways the texts use semantically laden letters are two of the areas considered. The texts are also scrutinized for evidence of confusing, ambiguous or misleading content in the area of the presentation and use of letters.

Tables included as Appendix D reveal the diversity of approaches taken by the texts. They also indicate a disturbingly large quantity of texts that do little to develop an understanding of the continuous variation of a variable, that avoid two or more dimensional problems, and that are in some way confusing or misleading in their use of letters.

Discussed extensively in Chapter III is the texts' treatment of semantically laden letters. As indicated in that chapter, over 60% of the texts devote less than 25% of their exercises and problems to word problems and other problems that require some interpretation of the semantic meaning of a letter. Furthermore, the observation is made that for many of the texts, those few word problems that are included are carefully selected so as to fit into some algorithmic pattern or to in some other way overly simplify the problem. This has the effect of allowing students to avoid the struggle of understanding and relating mathematically to the semantic content of a problem. <u>This general paucity of semantically laden problems not only effects students' understanding of semantically laden letters but must effect students' general problem solving skills.</u>

Chapters IV and V of this dissertation deal with "matter learned" in that they are the result of an investigation into student conceptions about the use of letters in algebra. The latter topic is so broad as to make it impractical to thoroughly investigate it in one study. Thus, these chapters focus exclusively on student conceptions of semantically laden letters.

In Chapter IV, four stages are described that comprise the preliminary research to this investigation. In the first stage, clinical interviews were conducted on problems involving a translation from an English sentence to an algebraic expression. Evidence was found supporting a working hypothesis that students often view semantically laden letters as labels for objects rather than numbers.

In the second stage, a diagnostic test was given to over 150 sophomore and junior college math students on a version of one of the translation problems used in the first stage. This diagnostic test gave the students multiple choices for describing the meaning of the letters. Over 40% of the students answered incorrectly, most opting to choose a nominal label as the meaning of the letter. Dramatically, 22% of the students felt strongly enough about an incorrect response to the translation problem and interpretation of the letter as label that they incorrectly identified the letter S with the word "professor" and (presumably) the letter P with the word "students."

Chapter IV continues with a report on the third stage of the preliminary research. This third stage was another diagnostic test, again designed to test for students' interpretation of semantically laden letters. The results of this test seemed, at first, consistent with those of previous tests, though the error rate (77%) was much larger, possibly indicating a greater complexity to the problem. But a closer inspection of the results seemed to indicate that the working hypothesis, that students view semantically laden letters as labels for objects, was not adequate. Students, in addition to accepting an interpretation of the letter as label also attributed quantitative value to the letters and did so with more than one attribute. For example, one letter not only stood for the quantity, but also for the price of an object.

This finding was borne out in the fourth stage of the research, reported on in Chapter IV, which was comprised of clinical interviews of students solving the above problem. An edited transcript of one student's attempted solution is presented and analyzed demonstrating that students are capable of attributing several distinct meanings, both quantitative and qualitative, to one letter in a single problem context. <u>Students' understanding of the concept</u> (of letters) <u>appears to be tenuous, vague, unstable</u> and only sporadically quantitative.

The stage was now set for the final phase of the research, comprised of four stages which are reported on in Chapter V. The major goals of this phase of research was to develop a more refined hypothesis for describing student conceptions of semantically laden letters and to investigate the frequency of these faulty conceptions.

Chapter V begins with a description of the first stage which was comprised of the clinical interviews of nine college math students solving algebra word problems. The problems selected had two attributes of consideration, for example, price and quantity, which made careful definitions of letters more essential to the solution of the problem.

The transcripts of these interviews were analyzed in the second stage of the research, the result of which is a list of eleven categories of meanings that students were observed to attribute to letters, two of which are at times appropriate, the rest of which are not. In addition, it is noted that, though students seem to shift the meaning of a letter from one categorization to another, they do not attend to those distinctions that allow for those different categorizations; that for them the letters have a more global, amorphous, less differentiated meaning that in some way encompasses several of what should be distinct categorizations. This leads to the hypothesis, described in Chapter V, that students view referents of semantically laden letters to be <u>undifferentiated</u> <u>conglomerates</u>, identifying the letter with an entire, complex, overly generalized referent rather than a particular attribute of that referent.

Additional evidence for this was found during the refined analysis of the interview transcripts conducted during the third stage of the research. This evidence took the form of multiple occurrences of the four behaviors described in Chapter V and repeated here for reference.

A. The student does not give clear, stable, quantitative definitions for the letters s/he uses.

B. The student evidences a use of letters that would fit either category VII, VIII, or XI above, and/or interprets letters in a way consistent with more than one of the remaining categories, indicating multiple or shifting meanings for the letter.

C. the student accepts the juxtaposition of two contradictory quantitative uses of the same letter.

D. The student is unable to drop part of the semantic load that a letter carries even when it is necessary and appropriate to do so.

A significant portion of Chapter V consists of segments of interview transcripts with extensive analyses that illustrate these four behaviors. Also in Chapter V is a description of a somewhat arbitrary, though, conservative standard which states that if a student demonstrates three of the four behaviors in the solution of a problem, it can be concluded that that student views semantically laden letters as undifferentiated conglomerates. According to that standard, seven of the nine students interviewed did so view semantically laden letters.

Finally, Chapter V presents the results of two written diagnostic tests given to a total of 254 college calculus students. The goals of those tests were to obtain data on the overall success rate on the types of problems given in clinical interviews and to see whether a correlation exists between evidence for the four behaviors and success or failure on the problems.

Tables presented in Chapter V indicate that, overall, less than 50% of the problems were answered correctly and that, on an average, students got slightly fewer than one out of two problems correct with 34% of the students getting no problems correct. As for the second goal of correlating the four behaviors with success or failure on the test, it is indicated that 87% of the 245 students who used any algebra in the solution of the problems exhibited behavior A and that those students scored significantly worse than students who gave a proper definition for the letters they used. Furthermore, those students who exhibited at least two of the four behaviors in at least one of the two problems they solved, scored significantly worse than the remaining students.

This last result, as indicated in Chapter V, is problematic because evidence for three of the four behaviors is somewhat dependent on verbal input. Written diagnostic data were found to be less revealing of students' conceptions (and more ambiguous) than clinical interview data. Because of this, the written tests did not provide solid evidence supporting the hypothesis that students view semantically laden letters to be undifferentiated conglomerates but the data obtained are at least consistent with such an hypothesis.

Finally, in Chapter VI, is a discussion of some of the myriad implications and potential research questions for the future that stem, respectively, from Chapters II, III, and IV and V. The topics that are suggested for future research pertain to such things as student conceptions of other aspects of letter use beyond semantically laden letters, the testing of textbooks and curricular strategies in order to improve upon the way letters are taught, and an expanded investigation into other aspects of semantically laden letters not dealt with in this dissertation.

Furthermore, it is suggested that <u>students' tendency to only vaguely and amorphously define the letters they use is likely to have implications and ramifications not only in other courses and subjects where letters are used symbolically but also in their ability to understand various other abstract concepts.</u> Acceleration and energy are two examples.

Also in Chapter VI, the curricular custom of spending very little space and time helping the students understand algebraic letters in all of their complexity is questioned. <u>The tendency on the part of most mainstream texts to include relatively few semantically laden problems is also</u>

questioned. And it is suggested that students' difficulties with semantically laden letters are reflected in a larger difficulty with general problem solving and the ability to apply mathematical skills to real world situations. All of this suggests a more general pedagogical implication which argues against a curriculum that focuses primarily on the "basic skills" of the mechanical manipulation of equations and expressions that are devoid of semantic meaning.

The last section in Chapter VI is "grist for the cognitive science mill." Some interesting parallels between the definition of an undifferentiated conglomerate given in Chapter V and Vygotsky's definition of his most primitive precursor to a true concept which he calls a heap are noted. Specifically, the author argues that though students' thinking about letters in Algebra may be extremely primitive, that does not necessarily suggest that their thinking abilities in general are primitive, and that conclusion is supported with the work of Bamberger and Schön. Finally, it is suggested that if we could learn more about undifferentiated conglomerates, we would be making an important contribution to cognitive science.

Throughout Chapter VI, allusions are made to many of the limitations of this study. Most of these limitations have one thing in common; they pertain to the fact that the scope of this dissertation has been focused onto a highly specific piece of the mathematics curriculum. Each of the research questions suggested in Chapter VI are topics that might have been covered had the immediate goals of this dissertation been less focused. But to broaden the scope of a research study often means the loss of some of the finer details. Highly focused clinical interviews afford the investigator the ability to gain insights into student conceptions that a less specific and focused study might not.

The author is aware of the danger of going in the opposite direction and being so focused that the topic being studied is of little consequence to the field. One can make an analogy with the field of education and the number line. The number line is made up of an infinite number of points and has infinite measure, but each of those points has measure zero. It is important that one's research topic is not so focused that it has a measure of zero. One would hope, rather, that one's topic spans a significant measure.

This author believes that the topic of letters in precalculus Algebra, with a focus on semantically laden letters, does so. For as indicated in Chapter VI, the results of this study have implications that go beyond algebraic symbols and beyond the symbolization process in general, and into issues that relate to general problem solving skills and to cognitive science. And as mentioned in the beginning of this chapter, by learning about what those crucial and, for students, confusing mathematical concepts really mean, studying how they are taught, and, especially scrutinizing how students view them, we make way for the revitalization of the mathematics curriculum and prepare for a future when just about everyone will come close to being a mathematician.

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APPENDIX A

LEARNING WITHOUT UNDERSTANDING:

THE EFFECT OF TUTORING STRATEGIES ON ALGEBRA MISCONCEPTIONS

Presented in The Journal of Mathematical Behavior, Vol 3, No. 1, pp. 3-27, 1980.

LEARNING WITHOUT UNDERSTANDING:
THE EFFECT OF TUTORING STRATEGIES ON ALGEBRA MISCONCEPTIONS

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Research reported in this paper was supported by NSF Award No. SED 78-22043 in the Joint National Institute of Education- National Science Foundation Program of Research on Cognitive Proccesses and the Structure of Knowledge in Science and Mathematics.

ABSTRACT

In this paper we discuss some research on students' abilities to translate English word problems into algebraic equations. In particular we identify a common error pattern in very simple word problems, called the reversal error. Results are then described from a set of tutoring interviews in which an attempt was made to correct students" misconceptions of this kind. We conclude, based on these tutoring interviews, that for many students the reversal misconception is a resilient one which is not easily taught away. Although the surface behavior of the students changed, continued probing in the interviews revealed that many of the students' misconceptions remained unchanged. We believe these results underscore the importance of distinguishing between performance and understanding as outcomes of instruction.

INTRODUCTION

At universities across the country, more and more academic departments are requiring their students to take mathematics. The study of mathematics is no longer restricted to students in Engineering and the Physical Sciences. From Business to Forestry, from Hotel and Restaurant Management to Nursing, students are required to take at least precalculus math, and often calculus and statistics. In the Spring of 1980, over 4,500 people were enrolled in precalculus and service calculus courses at the University of Massachusetts. (This is in addition to students in Engineering or the Physical Sciences who take more rigorous calculus courses.) 4,500 represents approximately 25% of the undergraduate enrollment. Traditionally, this figure is higher in the Fall.

What kind of learning is taking place for these 4,500 students? How well prepared are they to apply the large number of manipulative skills they have acquired to their fields of interest? Most of these students can solve quadratics, manipulate equations, find derivatives, and pass exams, but how do they fare at the interface between mathematical symbols and verbal descriptions of real world problems?

SOME BASIC MISCONCEPTIONS

To try to answer some of these questions we have been testing and interviewing students on problems that require them to translate from one symbol system to another. In some tasks, we ask students to translate an English sentence to an algebraic equation, or vice versa. In others, we ask students to interpret information in tabular, graphic, or pictorial form into an algebraic equation. The results of these studies indicate that many students fare poorly at these translation activities. The following two problems are among those we have given on diagnoistic tests:

> Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.
>
> Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel." Let C represent the number of cheesecakes ordered and let S represents the number of strudels ordered.

In a group of 150 first-year Engineering majors, only 63% were able to answer the Students and Professors problem correctly, and only 27% answered the Cheesecake problem correctly (see Kaput and Clement, 1980; and Clement, Lochhead, and Monk, 1979). There was a very strong pattern in the errors on these problems: two thirds of the errors in both cases took the form of a reversed equation, such as 6S=P or 4C=5S. Further results indicate that students in the social sciences do considerably worse on these questions as would be expected. (In preliminary tests, only 43% of these students answered the Students and Professors problem correctly.)

Initially, it was thought that these mistakes were simply careless misinterpretations due to specific wording of the problem. However, the reversal is also quite common in problems which call for translations from pictures to equations or data tables to equations. This suggests that the reversal error is not primarily due to the specific wording used in the word problem. In addition, lengthy video-taped interviews with students have indicated that the difficulty is quite persistent in many cases.

Some students appeared to use a word order matching strategy by simply writting down the symbols 6S=P in the same order as the corresponding words in the text. Others, however, demonstrated a general semantic understanding of the problem (i.e., that there are, in fact, more students than professors), yet they persist in writing reversed equations. The interviews reveal disturbing difficulties in the students' conceptualization of the basic ideas of equation and variable. For example, some students have what we call a figurative concept of equation, i.e., that since there are more students than professors, the coefficient, 6, by virtue of the fact that it is bigger than 1, should be associated with the "bigger variable," S. This results in the reversed equation, 6S=P. (See Clement, 1980).

Other students have explicitly demonstrated erroneous and/or unstable concepts of variable. Davis (1975), in analyzing the clinical interview of a 12-year-old solving an algebra problem, came to the conclusion that the student "...was not recognizing that x was, in fact, some number." Many of the college students that we have interviewed and tested have demonstrated that they too do not recognize the use of letters as standing for numbers. They confuse the use of letter as a variable with the use of letter as a label or unit. These students also tend to write the reversed equation 6S=P as the answer to the Students and Professors problem. When questioned, they read the equation as "six students for every professor" and directly identify the letter S as a label standing for "students" rather than as the proper, "number of students." Concomitant with this misconception of the use of letters in equations is a misconception of the use and meaning of the equal sign. Here, the equal sign apparently means "for every" or "is associated with" rather than "is numerically equal to."

PILOT TUTORING INTERVIEWS: EXAMPLES OF TRANSCRIPTS

After becoming convinced that the student's inability to do the Students and Professors problem and the Cheesecake problem is a significant difficulty, we decided to address the question of the resiliency of these misconceptions. At first we thought it might simply be a matter of pointing out the mistake to the students. In taped pilot interviews, different teaching strategies were tried, including:

1) Simply telling the student that the reversal is incorrect.
2) Telling the student that the variable should be thought of as "number of students," not "students."
3) Pointing out (with pictures) that since "students" are a bigger group than "professors," one must multiply the professors by six to create an equality.
4) Asking the students to test the equations by plugging in numbers.
5) Asking the students to draw graphs and/or tables.
6) Specifically showing the students how to set up a proportion to solve the problem.
7) Demonstrating a correct solution to the student, using an analogous problem.

This was done with nine students, most of whom had taken one semester of calculus, who had all initially reversed the Students and Professors problem. These interviews were fairly informal, in that teaching strategies were picked at the discretion of the interviewer in response to a perceived misconception. With most students, several strategies were tried as the problem persisted. The interviews lasted between 45 minutes and one hour, and covered several problems of the type above.

Our main conclusion, based on these pilot interviews, was that the reversal problem is a resilient one and that the students' misconceptions pertaining to equation and variable are not quickly "taught" away. In fact, at least seven out of the nine students demonstrated in one way or another that they maintained the reversal misconception.

Dawn, for example, initially reversed the Students and Professors problem. During a session lasting more than 20 minutes, she was alternately "taught" and then interviewed. Several teaching strategies were used, including the use of tables, the focus on variable as number, the techniques of plugging in numbers to test an equation, and others. Throughout, she made comments like "6P=S, that's weird. I can't think of it that way." She claimed that the interviewer was "shaking all her foundations." Eventually, however, she agreed that 6P=S was correct and was able to translate her learning to the Oil and Vinegar problem which follows. But then, she spontaneously redefined the problem to fit her preconceived notion, as seen in the following transcript segment:

(The Oil and Vinegar problem asks for an equation which represents the fact that there is 3 times as much oil as vinegar in a salad dressing.

S: (Draws a table showing 3, 6 and 9 under O, and 1, 2, and 3 under V.) ...my first impulse would be to write three times; three times O equals V.

I: MmMm.

S: So then, because that's wrong, I would change it to 3V=O because I know it's the other way around... So then I'm gonna plug that in. And that's right. (Writes 3V=O).

I: Well, why don't you-; I-I'll.

S: I know that's right [3V=O] because I make oil and vinegar dressing. And if you had a-if you had a cup--- I'm rationalizing this---if you had a-

I: Okay.

S: a-if you had a cup of oil and vinegar (draws cups) you'd put this much oil in and that much-; I mean this much vinegar in and that much oil or else it would be really greasy...

(Dawn drew the following pictures. She pointed to the one on the left when indicating vinegar.)

(vinegar) (oil)

Fig. 1

I: So you're saying that what this equation is saying is you're putting in more vinegar than oil? (Indicates equation 3V=O).

S: Uh-huh.

What is striking about this example is that when Dawn first wrote 3V=O she knew that there was more oil than vinegar. However, on reexamining the equation, her reversal misconception apparently took over, causing her to lose sight of the original relationship.

Don, another precalculus student, had a similar experience. He too, struggled through 30 minutes of work on the Students and Professors problem, where the interviewer tried several teaching strategies. One technique that he found helpful was graphing, and he applied this technique to the Goats and Cows problem, which reads as follows:

> "Write an equation using the variables G and C to express the fact that on a certain farm there are five times as many goats as cows. Let G stand for the number of goats and C for the number of cows."

Fig. 2a.

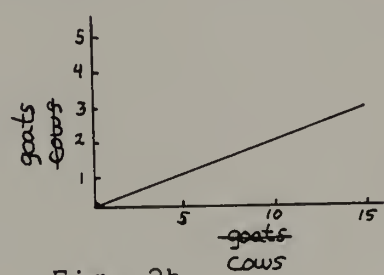

Fig. 2b.

The graph in Fig. 2a. was appropriate, indicating that there were 5 times as many goats as cows. He then wrote the correct equation 1G=5C, while referring to his graph. However, he read it as follows: "1 goat equals 5 cows." In analyzing the equation further he said that "5 goats=25 cows," having derived this from the equation by multiplying both sides by five. A complete shift had occurred. The interviewer then asked if his equation was consistent with the graph. The following ensued:

S: If you had 1 goat, you'd have to have 5 cows. If you
had 5 goats, you'd have to have 25 cows.

I: And does this equation express that?

S: Yup

I: ...does this graph express that?

S: Mmm--(6 sec)--Nope--the goats and cows should be on the
other side-- so it should be the number of goats on the
bottom, I mean number of cows--

I: Uh-huh.

(Don is in the process of making the vertical axis "goats" instead of "cows" and making the horizontal axis "cows" instead of "goats." In so doing, his graph becomes incorrect.) (See fig. 2b.)

S:...so that should be-- that should be the cows here, and that should be goats on this si-; on the...and this should

be the cows here---and then it'd be alright.

I: Feel comfortable with that?

S: Yup

Don has totally lost sight of the original problem. Both he and Dawn learned to write the correct equation but then reversed the meaning of the original problem.

Peter, a calculus and physics student, also learned to write the correct equation for the China problem, which states that there are 8 times as many people in China as there are in England. He, unlike Don and Dawn, was able to stay conscious of the original meaning of the problem. However, after 45 minutes, it became apparent that though his behavior had changed, i.e., he was able to write the correct equation, he still tenaciously retained his original misconceptions about variables and equations, as the following section of transcript illustrates. Peter refers to the correct equation for the China problem, 8E=C, to help him understand analogous problems.

S: I mean I feel confident here now.

I: Okay. Um--

S: What's that-; I love that example where-; well the one with China, alright. (I'll) do that one again. So you have the ratio China..to..uh, England was equal to 8 over 1.. so that's--

I: Mmmm.

S: You know, that's how-; I should know that.. is that China is gonna equal 8E. Just in a pure algebraically sense. But I don't think of it that way. I think there's 8 little-; 1 Chinese person for 8 little.. you know what I mean?

I: So what does the C mean there? (Points to the equation 8E=C).

S: C means the English person.. uh, England itself.

I: Uh-huh.

S: The number of people in England...naw, that's wrong..uh,yeah. The number of people in England, I'd say.

I: And what does the E mean?

S: The number of people in China.

Peter's misconceptions are so resilient that he is willing to associate the letter E with China and the letter C with England rather than change his internal conceptualization.

Dennis is another student who, like Peter, tried hard to internalize the teaching strategies that he received. One of those strategies taught that since the group of professors is smaller than the group of students, it is the P that needs to be multiplied to achieve equality. He applied this strategy to get the correct answer on the Students and Professors problem and also on an analogous problem. Nevertheless, upon reading the Cheesecake problem, he immediately reversed the equation. He wrote 4C=5S, which he read, "four cheesecakes equals five strudels." He validated this equation by showing that if you multiply both sides of the equation by five, it gives you the relationship that there are 20 cheesecakes for every 25 strudels. When questioned further, he attempted to apply the above teaching strategy to the problem, which led to the following discussion:

S: Well, there's the strudels...(9 sec)...and this will be the cheesecakes. And uh, overall they occupy the same area. The cheesecakes are just bigger... 1 cheesecake is bigger than 1 strudel.

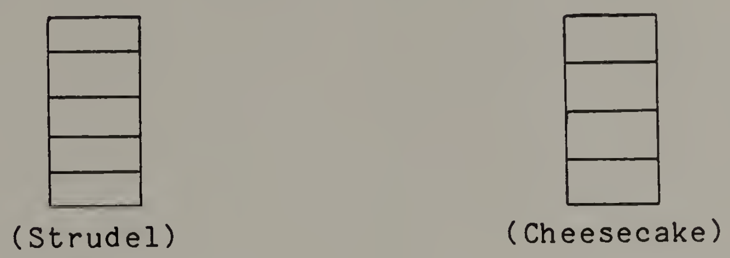

Fig. 3

(Dennis drew the above picture where the figure on the left represents strudel and the one on the right, cheesecake.)

I: I see. Is there any way that that can be expressed-- the idea that each group of cheesecake is bigger than each group of strudel--by either this equation or some other equation?

S: Umm...well,you could say...one S equals a fractional amount of cheesecake 'cause it's not the complete thing. So if you say.. ..um..I don't know what that particular fraction would be... (16 sec)...well, say 8...(5 sec)...

I: And what did y-; you wrote down 80?

S: Point 8. (.8).

I: Oh, point 8.

S: (unintelligible - three words) like 80%.

I: I see. Uh-huh.

S: 1 strudel is 80% the size of 1 cheesecake.

(Dennis then experiments with different equations and ratios, acknowledging the importance of the fraction 4/5ths.)

I: Somehow you knew it had something to do with--you said the fraction 4/5ths--how did you know that? Where did--?

S: Well, that was just by using the idea that--these are just larger than these; a larger value.

I: You're pointing to the cheesecake?

S: Well the cheesecakes are larger than the strudels.

I: Larger than the strudels.

S: Yes... okay. So it says 4 cheesecakes and 5 strudels--that's a 4 to 5 ratio. That's where I came up with this point 8.

I: Uh-huh.

S: For an individual strudel being 80% the sixe of an individual cheesecake.

I: Now I-I-I'm getting confused again. When you're saying 'individual strudel' what do you mean?

S: Well, if you were to take one--instead of messing with all these 5 and 4 of these things, you just pick 1 of each--

I: I see.

S: --and look at 'em, you'll see that the strudel's only 80% as big as the cheesecake.. from this ratio here...

I: Uh-huh.

S: ..this 4 to 5 ratio.

I: A-as big in terms of--?

S: Physical size.

I: I see...okay.

S: Or you could use mass or whatever unit you want to define it by.

Dennis appears to have confused the notion of numerical or cardinal size with physical size. The problem explicitly states that C stands for the number of cheesecakes. The purpose of the instruction he received was to teach him to be aware of the relative size of the different groups in terms of their cardinality. Instead, he has focused on their relative size in terms of physical properties. This causes him to write an equation that is the reverse of the correct one.

Not all of the errors and misconceptions were as blatant as the preceeding ones. However, if a student says, "I know how to get the right equation but it looks weird to me," or if a student reads the correct equation 6P=S as 6 professors for every student, we have concluded that the student does not truely understand the process of writing an equation from an English sentence. We now believe that writing a correct equation does not necessarily always imply understanding.

The following is a summary of the criteria we used in judging whether a student demonstrated a lack of conceptual understanding. These criteria helped us to distinguish the students who write the correct answer without understanding from those students who truly understand the problem. We concluded that a subject has demonstrated a lack of conceptual understanding of the problem if:

1. S/he remained incapable of writing the correct answer throughout the interview.
2. After correcting the reversal mistake, s/he at a later time:

   a) reverts back to the reversed equation.

   b) accepts the correct equation but reverses the meaning of the original problem. c) accepts the correct equation but switches the meaning of the letters (i.e., S stands for professor).

   d) identifies the correct equation as being "weird" or "not making sense".

e) acknowledges that the correct equation "works" but states that s/he doesn't know why it works.

3. The student reads the correct equation erroneously (e.g., S=6P is read "one student for every six professors").
4. After making a minor arithmetic mistake while checking the correct equation, s/he immediately doubts and discards the correct solution before rechecking the arithmetic (i.e., his/her belief in the correct equation is extremely tenuous).
5. The student demonstrates a clear misunderstanding of the use and meaning of letters in equations (e.g., by being unable to replace the letter with appropriate values).
6. After apparently learning how to solve more elementary problems, the student:
   a) makes no attempt to apply his/her learning to a more difficult problem.
   b) does attempt to apply his/her learning but does so erroneously.

At least seven of the nine students in our pilot study demonstrated a lack of conceptual understanding in terms of the above criteria.

FOLLOW UP STUDY USING STANDARDIZED TUTORING INTERVIEWS

On the basis of the initial set of nine interviews we became convinced that the reversal error and other related errors cannot be corrected by simply demonstrating the correct solution or by explaining to the student why his answer was wrong. They do not appear to be casual or careless mistakes that mere concentration can eliminate. Rather, they appear to be caused by deeply ingrained and resilient misconceptions.

To further test the hypothesis that the misconceptions are resilient we designed a more systematic teaching strategy. We wanted to give the students a written unit which contained a clear demonstration of how to do these problems and which had an explicit technique which the students could learn. This unit focused on the idea that letters in equations are variables that are meant to be replaced with appropriate numbers. This allows one to test whether an equation is an appropriate one. (this teaching unit appears in the Appendix.) The teaching unit is by no means our ideal approach to instruction. What we were interested in knowing was whether a fairly simple, traditional, algorithmic approach to teaching would be sufficient to help the students with the reversal error.

Whereas the nine students previously interviewed were drawn from various introductory math and physics courses, the six students to who we gave the standardized unit were enrolled in the first year of a rigorous calculus course designed for Engineers, Scientists, and Math majors. All six had reversed the equation for the Students and

Professors problem on a written diagnostic test. These six students were interviewed and taped as they were working on the teaching unit. They were asked to "think out loud" as then worked their way through the various explanations and practice problems in the unit. Their performance on each problem was then graded in one of the following three ways: initially correct; initially incorrect but eventually correct (usually with prompting from the interviewer); and incorrect. Prompting took the form of reminding the students about the teaching strategy and/or asking them to check their answers. The interviewer usually had the student work on each problem until it was correct.

The results are shown in Table 4. These results might lead one to believe that some significant learning occurred. After all, by the time students reached the Sandwich problem, four out of five initially got it right; and though four students initially erred on the Cheesecake problem, all eventually worked to the correct answer.

However, our conclusion is just the opposite. We have concluded that in at least five out of the six cases, significant learning did not occur. Though students' behavior for the most part was changed, we believe that their conceptual understanding of equation and variable remained, for the most part, unchanged.

	Initially Correct	Initially Incorrect but then Correct	Incorrect
<u>England Problem</u> (analogous to Students and Professors)	4	2	
<u>Goats</u> (also analogous, with different wording)	4	2	
<u>Council</u> (analogous to Students and Professors, but additive)		5*	1
<u>Cheesecake</u>	2	4	
<u>Sandwiches</u> (analogous to Cheesecake)	4	1	

*Data on the Concil problem is somewhat tangential to our discussion of reversals because most of the errors there had to do with inappropriately assuming that the problem was multiplicative, an error that was not addressed by the teaching strategy.

Table 4.

EXAMPLES FROM STANDARDIZED TUTORING INTERVIEWS

Deirdre was able to write down the correct equations, but the following statement is evidence that her learning was merely procedural and not conceptual. She had just written a correct equation and said "this is probably right because it works. It works (by plugging in values) but I don't know why it works." Later, she said, "It works but I don,t think it works."(!)

Carol similarly has acquired the procedure but makes the comment that the correct answer "is not what you would immediately write down but the opposite." Carol, later in the interview, goes back to the Students and Professors problem and looks at the correct equation, S=6P, and tries to read what it says. "For every student there are...no, for every... see,it's not for every student there are 6 professors... I don't know. I'm confused now." An appropriate way to read S=6P is "the number of students equals six times the number of professors." To know that the letters stand for numbers is an essential component to understanding the problem and was the main goal of the teaching unit. However, that is still a very elusive idea for Carol.

Further evidence pointing to the fact that the learning that has occurred is not on a solid footing and is not backed by conceptual understanding was provided by two other students. Both, in checking their answers, made minor arithmetic mistakes. Rather than double-check their arithmetic, their response was to scuttle their correct equations and try different equations that were just stabs in the dark. Mona, for example, had the correct equation S/7=H/9 for one problem, but when it didn't check out she tried S/7=9H. She could provide no logical justification for this last equation. That she had no qualms about abandoning the original equation suggests that she had little conceptual understanding of it. Mona also demonstrated confusion by concluding that 4C=5S is incorrect for the Cheesecake problem (which it is), by plugging in 2 for both the C and the S. When asked what the 2 stood for, she said:

> "That would be the uh--the number of strudels--um, this equation doesn't really fit it--. For every 4 people who ordered cheesecake, 5 who ordered strudel. Say there's a group of --2 times 4, which is eight. For every group of 4-- for every group of 4 equals-- I guess C is like how many groups of 4 there are and S is, would be how many groups of 5."

Mona apparently does not understand the way letters are used as variables in an equation.

David demonstrates confusion in several different ways. He, on occassion, plugs numbers in incorrectly. He makes statements like, "It works but it's wrong." He also becomes confused when he tries to read algebraic sentences, as shown in the following transcript excerpt.

I: Is that right? (S=5/4C)

S: Yeah.

I: Read this to me. I'm pointing to S=5/4C. What does that say?

S: Alright, there's 5 strudels for every...5 strudels is equal to 1 and 1/4 cheesecakes. I don't know--how did I get-; I had the other one backwards---

I: 5 strudels. Now where-where do you see 5?

S: I mean 1 strudel.

I: 1 strudel is equal to 1 and 1/4 cheesecakes?

S: Yeah. I was looking up here (at the problem statement) just (jumbled)-; I was reading this over while I said it. I did it backwards the first time.

I: Which-which is there more of, strudel or cheesecake?

S: Strudel.

I: Okay. So you said one strudel is equal to 1 and 1/4 cheesecake.

S: Mm, yeah...(8 sec)...

I: Hmm. Confusing isn't it?

S: Yeah.

I: What are you thinking right now?

S: I'm wondering why this one-; this way here works (points to S=5/4C).

I: You don't think it should work?

S: Right.

(David proceeds to check the correct equation S=5/4C by appropriately plugging in numbers. Still confused, he checks the opposite equation C=5/4S with the same numbers and finds it is incorrect. But S=5/4C doesn't "read" correctly for him, so he continues to play with numbers for a long time. He finally decides that S=5/4C is correct.)

S: ... this one (S=5/4CO. I'd say- I,d say I,d stay with this one if I had to.

I: Okay. Read this one-to me again.

S: ...No, maybe not.

I: What-

S: If I read it to you, it seems wrong.

I: Okay.

S: If I say-

I: Read it to me then.

S: 1 strudel-

I: Uh-huh.

S: --is equal to 1 and 1/4 cheesecakes.

I: Uh-huh. And that seems wrong?

S: Yeah. Because it's--; 1 cheesecake is equal to 1 and 1/4 strudel like I had down here (C=5/4S) but this equation doesn't work.

(After this, David again plugs in numbers and after a good deal of time says:)

S: So I'd say (S=5/4C) is right.

I: ...could you read that for me?

S: Um.. one strudel is equal to...S is equal to 5/4ths strudels.. equal to 5/4ths the cheesecake..it doesn't look like it works but it does.

David has finally learned to plug in numbers correctly and on the basis of that, decides on the correct equation. But he does so in a conceptual vacuum; more accurately, he does so with an incorrect conceptual framework that is resistant to change.

In analyzing the transcripts of the interviews with the six students. we were confronted with the difficult problem of judging whether the student conceptually understood each problem. We did this by categorizing the students' performance on each problem in one of the following three ways:

1. The student demonstrated a conceptual understanding of the problem, i.e., in the course of the execution and discussion of the problem, the student indicated that s/he understood that variables stand for numbers rather than individual objects, and that a larger coefficient is associated with the variable that represents the smaller group in order to equalize both sides of the equation.
2. The student demonstrated a lack of conceptual understanding of the problem. Criteria 1 through 5 of our previously discussed criteria for judging conceptual understanding were used to judge whether a student's solution to a particular problem should be categorized in this way.
3. Neither a conceptual understanding nor a lack of conceptual understanding of the problem were

demonstrated, i.e., it is unknown whether or not the student understands the problem.

This category, for the most part, was the modal one. However, it is our impression that the majority of subjects in this category would have been classified in category two if we had probed more deeply. Support for this view is provided by the data for the Cheesecake problem. Here, the students worked on and discussed the problem for the longest amount of time; time enough for the misconceptions to be revealed. Though all six students eventually wrote down the correct equation, five of them clearly demonstrated that they retained serious doubts and misconceptions about the problem. Their behavior was changed but their misconceptions remained. The results of this analysis appear in Table 5. (Compare Table 4.)

	Conceptual Understanding Demonstrated	Lack of Conceptual Understanding Demonstrated	Neither Conceptual Understanding Nor Lack of it were Demonstrated
England	1	1	4
Goats	1	2	3
Council		2	4
Cheesecake		5	1
Sandwiches		2	3

Table 5.

These results confirm the fact that the misconceptions students possess pertaining to variable and equation are deep seated and resistant to change. The results also underscore the fact that the ability to learn procedural techniques for solving problems does not entail an understanding of the essence of these problems. In this case, students' ability to write down the correct answer to a problem is a poor indicator of whether or not they understand what they are doing.

CONCLUSION

An important question remains: how can students learn to solve these problems with understanding? We believe that one answer to this question is that the fundamental concepts of variable and equation should not be treated lightly in high schools and colleges, nor should we assume that our students will develop the appropriate conceptions by osmosis. We also believe that the answer lies in encouraging students to view equations in an operative way--that equations represent active operations on variables that create an equality. This contrasts with the view of an equation as a static statement, where the larger coefficient is associated simplistically and incorrectly with the larger variable. Furthermore, we believe that it is essential that students be able to view variables as standing for number. Simple as it may seem, this last conception is a fairly abstract one and, for that reason, a very difficult one to teach. The development of specific teaching strategies that would adequately address these issues is an important task in need of further investigation.

Several members of our research group are finding, in pilot studies, that students' misconceptions are not limited to the reversal of equations, but that there are a number of other deep seated misconceptions pertaining to semantic aspects of algebra. The implications of these and the present study are that more attention must be paid to conceptual development in mathematics education. The level of mathematical incompetence of these students is evidence for the shortcomings of an educational system that focuses primarily on manipulative skills. That many students can succeed in a curriculum to the point of becoming Engineering and Science students, yet somehow have missed the mathematically essential notions of equation and/or variable is disturbing. That so many science oriented students are confused at the interface between algebraic symbols and their meaning is also disturbing. It suggests that an even larger proportion of non-science students are not gaining the skills that would be helpful in their careers. It also suggest that large numbers of students may be slipping through their education with good grades and little learning.

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APPENDIX I (to Appendix A):
TEACHING UNIT GIVEN TO STUDENTS
IN THE SECOND PART OF THE STUDY

We reproduce here the written material handed to the students for the remedial teaching part of the study.

Writing Algebraic Equations

Writing an algebraic equation from an English sentence is a deceptive task. We have found that a surprising number of people become confused when trying to write these equations. For that reason, we have developed a small unit for learning this skill. We ask that you follow the outline of the unit carefully.

In this unit, we ask you to go through three steps in writing an equation. THE THIRD STEP IS IMPORTANT!

Step 1. Understand the English sentence and describe what is asked for in your own words. Find numbers that would fit the relationship.

Step 2. Attempt to write an equation.

Step 3. CHECK YOUR ANSWER in the following way: REPLACE the letters in your equation with the numbers you found in Step 1 and see if both sides of the equation really are equal. If not, repeat Step 2.

On the next page is an example, using these three steps.

Page 2:

Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

Step 1. What this means to me is that there are more students than professors: specifically, 6 times more. So if there were 2 professors, there would be 12 students. If there were 10 professors, there would be 60 students.

Step 2. Attempt an equation. I'll try 6S=P.

Step 3. Check by replacing the letters with numbers from Step 1. I said 2 professors and 12 students. I replace S with 12 and P with 2 to get: 6(12)=2. THIS IS NOT TRUE. So, I will attempt another equation.

Step 2. 6P=S.

Step 3. Replacing S with 12 and P with 2, I get: 6(2)=12, which is true. So 6P=S is the correct equation.

Now you try one.

Page 3:

Write an equation to represent the following statement: "There are 8 times as many people in China as there are in England." Let C be the population of China. Let E be the population of England.

Please go through all three steps.

Page 4:

Write an equation using the variables G and C to represent the following statement: "On a nearby farm, the number of goats is five times the number of cows." Use G for the number of goats and C for the number of cows.

Don't forget to check!!

Page 5:

Write an equation to represent the following statement: "A certain council has 9 more men than women on it." Use M for the number of men and W for the number of women in your equation.

Don't forget to check by replacing letters with numbers.

Page 6:

Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who ordered cheesecake, there were five who ordered strudel." Let C represent the number of cheesecakes ordered and let S represent the number of strudels ordered.

Don't forget to check.

Page 7:

Given the following statement: "At the last football game, for every seven people who bought sandwiches, there were nine who bought hamburgers." Write an equation which represents the above statement, using S for the number of people who bought sandwiches and H for the number of people who bought hamburgers.

APPENDIX B

S STANDS FOR PROFESSOR:

SOME MISCONCEPTIONS CONCERNING THE CONCEPT OF VARIABLE

An abridged version of this paper appears in The Mathematics Teacher, Vol 74, No. 6, September, 1981 under the title, "Some Misconceptions Concerning the Concept of Variable."

S STANDS FOR PROFESSOR:

SOME MISCONCEPTIONS CONCERNING THE CONCEPT OF VARIABLE*

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Research reported in this paper was supported by NSF Award No. SED78-22043 in the Joint National Institute of Education-National Science Foundation Program of Research on Cognitive Processes and the Structure of Knowledge in Science and Mathematics.

*I would like to acknowledge the influence, suggestions, help in editing, and support of Dr. John Clement, whose extensive work in this area provided the foundation for this article.

Unfamiliarity Breeds Contempt

Letters in equations were my initial mathematical downfall. For me, it wasn't the x's, y's, and z's of algebra that caused the trouble. Nor was it the ε's and δ's of calculus, though they certainly caused some very panicky moments. It was my first sight of the α's and β's of abstract algebra that pushed me into a frenzied retreat from mathematics. At that time (ten years ago) I identified math as being little more than obscurity by abstraction and dehumanization by symbolization.

I have since returned to math, tackling α's, β's, and γ's left and right. (I even overcame ξ, a symbol that not only had a referent that was extremely difficult to understand, but was also impossible to write.) But, the memory of the difficulties I had has increased my interest in the problems associated with learning about mathematics and its symbols.

Much has been written in the last decade concerning the reasons why so many students have difficulty with and an aversion to mathematics. Compelling arguments can be presented about cultural, political, and psychological factors that have caused the mathematical downfall of many a would-be mathematician, scientist, business person, and computer programmer. As valid as these arguments may be, we must not neglect the inherent difficulty of the subject of mathematics itself. The curriculum for mathematics, from elementary school to graduate school, follows a path of increasing abstraction. As the curriculum becomes more abstract, the symbols used become more obscure. For many students, as was true for myself, unfamiliarity with mathematical symbols and the abstract concepts to which they refer breeds contempt for mathematics.

Notation: The Math Neophyte's Nightmare

In this paper, I will discuss a study that underscores the extent to which our students do not understand the use of letters in equations. This study is an extension of a body of research done by our group at the University of Massachusetts which has focused on students' ability to translate English sentences into algebraic expressions and vice versa. The results of previous research, based on diagnostic tests, video taped interviews, and the microanalysis of those interviews, indicate that students (including, for the most part, first year engineering students and/or students who have taken one or two semesters of Calculus) have a very difficult time with these translations.(See Clement, Lochhead and Monk (1981); Clement and Kaput (1979); and Clement (1980)). In addition, we have found that the misconceptions that students have surrounding the use of letters in equations contributes significantly to this difficulty.

One of the problems upon which much of our research has been based is the Students and Professors problem. It reads as follows:

> Write an equation, using the variables S and P to represent the following statement: "At this university there are six times as many students as professors." Use S for the number of students and P for the number of professors.

Fully 37% of a group of 150 entering engineering students at the University of Massachusetts were unable to write the correct equation, S=6P, in any form. The most common error was the reversed equation, 6S=P. (See Clement, Lochhead and Monk (1981))

Another problem we have used is the Cheesecake problem. It asks the student to write an equation that represents the fact that at a certain restaurant, for every four cheesecakes that were sold, five strudels were sold. More than 73% of the engineering students answered this question incorrectly, with the most common mistake being the reversed equation 4C=5S. Furthermore, students in those math classes that are oriented more towards business and the social sciences did appreciably worse on these problems.

We have proposed several possible explanations for this reversal error. One of those explanations is that students have a great deal of difficulty understanding the role and meaning of letters in equations. We have several taped interviews that support the belief that many students who write 6S=P believe that S is a label standing for "students" rather than a variable standing for "number of students." They will read the equation 6S=P as "there are six students for every one professor" pointing to the S as they say students and P as they say professors. Conversely, they will read S=6P as "one student for every six professors" instead of the appropriate "the number of students is equal to six times the number of professors."

This conception of viewing letters as standing for labels may have been implanted at a relatively early age. In elementary school and in the beginning of algebra, letters are often viewed as little more than labels symbolizing concrete entities. The sentence "two apples plus three apples is equal to five apples" can be symbolized as 2a+3a=5a. Similarly, 2x+3x=5x can mean two concrete things plus three more of those things equals five of those things. For example, it can mean two xylophones plus three xylophones equals five xylophones, or it can mean two "x's" plus three "x's" equals five "x's."

An important step in the development of the conception of the use of letters in equations is the recognition that a letter can stand for number. Here, 2x+3x=5x means, two times the number of xylophones on hand plus three times that number of xylophones is equal to five times that number of xylophones. Even more abstractly, it could mean "two times some number 'x' plus three times that number, is equal to five times that number." It is important to note that in the conditional equation 2a+4=5a, 'a' can no longer mean apples. The sentence "two apples plus four equals five apples" doesn't make sense. It must be viewed as two times some number of apples plus four is equal to five times that number of apples. All of this sounds obvious to the initiated, but the point is that it is apparently not at all obvious to the students.

S Stands for Professor

I am presently teaching a statistics course for business majors. Though most students have had two semesters of calculus, no calculus is used in the course. I was about to start a unit on random variables in probability when it dawned on me that the capital X of random variables was a symbol that I myself had had a very hard time understanding. Knowing that my students probably had a very weak notion of what a regular variable was, how could I expect them to understand random variables? I decided that, before diving into random variables, I would give them the following revised version of the Students and Professors problem to see whether or not they were confused about variables:

At this university, there are six times as many students as professors. This fact is represented by the equation $S=6P$.

A) In this equation, what does the letter P stand for?
   i) Professors
   ii) Professor
   iii) Number of professors
   iv) None of the above
   v) More than one of the above (if so, indicate which ones)
   vi) Don't know

B) What does the letter S stand for?
   i) Professor
   ii) Student
   iii) Students
   iv) Number of students
   v) None of the above
   vi) More than one of the above (if so, indicate which ones)
   vii) Don't know

The results were quite startling. Over 45% of the 33 students were incapable of picking "Number of professors" as the only appropriate answer in part A. Similarly, over 48% did not answer part B correctly. We believe that these results alone are significant and important. They support the hypothesis that students tend to view the use of letters in equations as labels that refer to concrete entities. P stands for "professor" or "professors," not the more abstract "number of professors." Furthermore, these results underscore the fact that students do have a great deal of difficulty with translations.

But there is more! If you notice, my choice of S standing for "professor" for answer i) in part B seems to border on the absurd. In fact, I must confess to having felt a little whimsical and capricious in including it as an option. Imagine my surprise when five people (over 15%) chose as their answer, "S stands for professor."

Critics might be quick to say that the students were pulling my leg. Regretfully, I am convinced that that is not the case. First of all, my students approached the problem as do most students answering any question having to do with mathematics, with no humor and with furrowed brows. Secondly, and more importantly, <u>every</u> person who chose "professor" for the answer in part B chose "none of the above" for their answer in part A. The latter is a consistent response in that those who would view S as standing for professor would also view P as standing for student. Since that option was not provided, they chose "none of the above".

Curious to see whether these results were more than an improbable quirk of human nature, we gave the same question to 119 students in a second semester calculus course designed for the social sciences. Though more students answered the question correctly, (61% and 58% for parts A and B, respectively) those that didn't were more inclined to pick "professor" for part B. Fully 24% of all the students said that S stands for professor. Furthermore, as before, every student who chose "professor" for part B chose "none of the above" for part A.

These results support a conclusion that we have believed for some time: that the tendency on the part of many students to write the reversed equation, 6S=P is not only a common one but is one that is deeply entrenched. In taped interviews, students have defended the choice of 6S=P over S=6P with strong logical explanations (though that logic is often based on misconceptions). Our hypothesis is that most students who believe S stands for professor also believe that 6 students= 1 professor (6S=P) is really the correct equation. And they believe it so strongly that, when presented with S=6P they assume the meanings of the letters have been switched. This result demonstrates that students can hold on quite tenaciously to misconceptions which "make sense" to them. It is an example of a misconception which is "resilient" in the face of factors which one would expect to change the student's mind. Such resilient misconceptions are unlikely to disappear in response to a quick lecture on the subject.

Implications: For the Love of Alpha

The implications for secondary school are important and suggest the following objective:

Students need to develop a better understanding of the basic concepts of variable and equation. More specifically, they should be able to distinguish between different ways in which letters can be used in equations. For example, they should know that letters can be used as labels referring to concrete entities (x stands for "xylophones"), or, alternately, as variables standing abstractly for some number or number of things (x stands for "number of xylophones"). The ability to make this distinction would prevent students from reading the equation S=6P as "one student for every six professors" or worse as "one professor for every six students," indicating the S for professor.

Having stated the objective, I recognize that reaching it is another matter altogether. I do not believe that there is a quick solution to the difficulty. It might even be the case that many secondary school students, and for that matter, college students, have not yet reached the necessary level of intellectual development to be able to make that distinction.

An important step, however, is that we, as math educators, should be aware of the distinction ourselves. In teaching word problems in an algebra class I have been, in the past, somewhat careless about writing "P=professors" rather than "P=number of professors." Now that I know what I know, I make every effort to say and write the latter and to verbalize the distinction.

As students progress from year to year in mathematics, the letters they use, like the concepts they are learning, become increasingly abstract, and, to them, ambiguous. If it is true that our students have a poor understanding of the use of letters in equations, what other central and crucial math concepts do they not understand? How much of the math work that they are doing is done in a conceptual void?

It is important that we stay aware of the difficulties that our students are having in trying to understand labels, variables, constants, parameters, and all of the rest of the uses of letters. It is equally important that we become aware of all of the other conceptual pitfalls to which our students can succumb. After all, if we can't help our students understand the x's and y's, they will never know the joy of understanding α's and β's. More importantly, they will drop out of or be inept in mathematics, a subject that has become a prerequisite for more and more careers in today's world.

Do you know how many of your students would say..."S stands for professor"?

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APPENDIX C

BIBLIOGRAPHY OF TEXTS REVIEWED IN CHAPTER III

BIBLIOGRAPHY OF TEXTS REVIEWED IN CHAPTER III

(listed by publisher)

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APPENDIX D

FOUR TABLES FROM THE TEXT BOOK REVIEW OF CHAPTER III.

(In this table, the numbers in the column on the left refer to texts that are listed in the Bibliography in Appendix C.)

a. REPLACEMENT SET	b. HOW VARIABLES VARY	c. AMBIGUITY
A. Sets of things B. Sets of numbers C. A number D. Unusual Approaches E. No definition	F. Unvaryingly G. Discretely H. Continuously	I. Contradictory usage J. Overuse of letters K. No definition

	a.					b.			c.		
JUNIOR HIGH TEXTS	A	B	C	D	E	F	G	H	I	J	K
2				x			x			x	x
8-9			x			x					
17-18	x					x				x	
21-22			x			x				x	
25-26		x					x				
31			x			x					x
32				x			x				x
33-34			x				x				x
TOTALS-JUNIOR HIGH	2	2	7	2	0	7	6	0	0	5	3
ALGEBRA I TEXTS											
4					x	x				x	x
10		x					x				
11			x				x				
14					x		x				
15			x				x			x	
16	x							x		x	
19			x			x					
27	x					x					x
28		x				x					
29		x				x				x	
30	x					x					
TOTALS-ALGEBRA I	3	3	3	0	2	6	4.5	.5	0	4	2

Table 6: Variables in Junior High and Algebra I texts.

(In this table, the numbers in the column on the left refer to texts that are listed in the Bibliography in Appendix C.)

a. REPLACEMENT SET	b. HOW VARIABLES VARY	c. AMBIGUITY
A. Sets of things B. Sets of numbers C. A number D. Unusual Approaches E. No definition	F. Unvaryingly G. Discretely H. Continuously	I. Contradictory usage J. Overuse of letters K. No definition

	a.					b.			c.		
ALGEBRA II TEXTS	A	B	C	D	E	F	G	H	I	J	K
3			x				x				x
5					x						x
12		x					x		x		x
13					x						x
20					x						x
TOTALS-ALGEBRA II	0	1	1	0	3	0	2	0	1	0	5
INNOVATIVE/EXPERIMENTAL TEXTS											
1		x				x			x		
6		x						x			
7			x				x				
24				x		x			x	x	
35		x					x			x	
36-37		x									
38	x							x	x	x	
39	x							x		x	
TOTALS-INNOVATIVE/EXPERIMENTAL	2	5	1	1	0	1	6.5	1.5	3	4	0
COLLEGE - PRECACULUS TEXTS											
23					x						x
40	x										x
41		x						x			
TOTALS - COLLEGE PRECALCULUS	1	1	0	0	1	0	.5	.5	0	0	2
GRAND TOTALS (n=41)	8	12		3	6	14		2.5	4		12
			12				19.5			13	

Table 7: Variables in Algebra II, innovative and college texts.

SEMANTIC CONTENT AND DIMENSION

(In this table, the numbers in the column on the left refer to texts that are listed in the Bibliography in Appendix C.)

A. Fewer than 25% of all problems were word or thought problems.
B. Word problems are introduced with pure number problems.
C. Wording of problems is contrived, avoiding mental effort.
D. Translations are done via key word match.
E. One variable problems are emphasized.

JUNIOR HIGH TEXTS	A	B	C	D	E
2					x
8-9	x		x	x	x
17-18	x	x	x		x
21-22	x		x		x
25-26			x		x
31			x		x
32					x
33-34				x	x
TOTALS-JUNIOR HIGH	6	2	9	4	13
ALGEBRA I TEXTS					
4	x	x		x	
10	x		x		
11	x		x		
14	x				
15		x	x	x	
16			x	x	
19	x	x	x		x
27	x		x		
28	x	x	x	x	
29	x	x	x	x	
30			x		
TOTALS-ALGEBRA I	8	5	9	5	1

Table 8: Semantic content and dimension in Junior High and Algebra I texts.

SEMANTIC CONTENT AND DIMENSION

(In this table, the numbers in the column on the left refer to texts that are listed in the Bibliography in Appendix C.)

A. Fewer than 25% of all problems were word or thought problems.
B. Word problems are introduced with pure number problems.
C. Wording of problems is contrived, avoiding mental effort.
D. Translations are done via key word match.
E. One variable problems are emphasized.

ALGEBRA II TEXTS	A	B	C	D	E
3	x	x	x	x	
5	x		x	x	
12	x	x		x	
13	x	x	x		
20	x				x
TOTALS-ALGEBRA II	5	3	3	3	1
INNOVATIVE/EXPERIMENTAL TEXTS					
1					
6					
7	x	x	x		x
24	x		x		x
35					x
36-37	x				x
38					
39					
TOTALS - INNOVATIVE/EXPERIMENTAL	4	1	2	0	5
COLLEGE - PRECACULUS TEXTS					
23	x				x
40	x				
41		x			x
TOTALS - COLLEGE PRECALCULUS					
GRAND TOTALS (n=41)	29	13	25	11	27

Table 9: Semantic content and dimension in Algebra II, innovative, and college texts.

APPENDIX E

CLINICAL INTERVIEW PROBLEMS

CLINICAL INTERVIEW PROBLEMS

Some of the Problems Given to Students During the Clinical Interviews.

(Not all students received all of the problems)

1. A boy bought a collection of frogs and turtles. The number of frogs that he bought was three times the number of turtles that he bought. Frogs cost 3 dollars each and turtles cost 6 dollars each. He spent $60.00 altogether. How many frogs did he buy?

(This problem was given to only two students, Ann and Beth).

2. I went to the store and bought the same number of books as records. Books cost two dollars each and records cost dix dollars each. I spent $40 altogether. Assuming that the equation 2B + 6R = 40 is correct, what is wrong, if anything, with the following reasoning. Be as detailed as possible.

2B + 6R = 40 Since B = R, I can write

2B + 6B = 40

8B = 40

This last equation says 8 books is equal to $40. So one book costs $5.

3. A person went to the store and bought pecans and cashews. He bought a total of 100 nuts. The number of pounds of pecans he bought was the same as the number of pounds of cashews. 8 pecans weight 1 pound and 12 cashews weigh 1 pound. How many pounds of pecans did he buy?

4. A person went shopping for books and records. He spent a total of $72. The price of each book was the same as the price of each record. He bought two books and six records. What is the price of one record?

5. A person went shopping for books and records. He spent a total of $72. The number of books bought was the same as the number of records bought. Books cost two dollars each and records cost six dollars each. How many records were bought?

APPENDIX F

FIRST WRITTEN DIAGNOSTIC TEST

FIRST WRITTEN DIAGNOSTIC TEST

(Students who took the written diagnostic test were given the following instructions)

Please answer the following two problems. SHOW ALL OF YOUR WORK. DO NOT ERASE. Just lightly cross out your mistakes. Your answers will be kept completely confidential and will in no way affect your grade in this course.

(Half of the students were given problems 1 and 4; the other half were given problems 2 and 3. On the original test, each problem was given on separate sheets.)

1. A Biology teacher bought a collection of frogs and turtles. The number of frogs that he bought was three times the number of turtles that he bought. Frogs cost 3 dollars each and turtles cost 6 dollars each. He spent $60.00 altogether. How many frogs did he buy?

2. A woman had a container of red and green blocks that weighed a total of 91 ounces. Red blocks weigh one ounce each. Green blocks weigh three ounces each. The number of red blocks was four times the number of green blocks. How many red blocks did she have?

3. A Biology teacher bought a collection of frogs and turtles. The price of one frog was three times the price of one turtle. He bought three frogs and six turtles. He spent $60.00 altogether. What is the cost of one frog?

4. A woman had a container of red and green blocks that weighed a total of 91 ounces. She had one red block and three green blocks. The weight of a red block is four times the weight of a green block. How much does one red block weigh?

APPENDIX G

SECOND WRITTEN DIAGNOSTIC TEST

SECOND WRITTEN DIAGNOSTIC TEST

(Students who took the written diagnostic test were given the same instructions given in Appendix F.)

(Half of the students were given problems 1 and 4; the other half were given problems two and three. On the original test, each problem was given on separate sheets.)

1. A biology teacher bought a collection of frogs and turtles. The number of frogs that he bought was four times the number of turtles that he bought. Frogs cost one dollar each and turtles cost three dollars each. He spent $91.00 altogether. How many frogs did he buy?

2. A woman had a container of red and green blocks that weighed a total of 60 ounces. Red blocks weigh three ounces each. Green blocks weigh six ounces each. The number of red blocks was three times the number of green blocks. How many red blocks did she have?

3. A biology teacher bought a collection of frogs and turtles. The price of one frog was four times the price of one turtle. He bought one frog and three turtles. He spent $91.00 altogether. What is the cost of one frog?

4. A woman had a container of red and green blocks that weighed a total of 60 ounces. She had three red blocks and six green blocks. The weight of a red block is three times the weight of a green block. How much does one red block weigh?

APPENDIX H

SUPPLEMENTARY STATISTICAL DATA FROM THE WRITTEN DIAGNOSTIC TESTS

SUPPLEMENTARY STATISTICAL DATA FROM THE WRITTEN DIAGNOSTIC TESTS

In Chapter V, two written diagnostic test were described and some results were reported. Some other results of those test that are considered to be tangential to the body of the dissertation will be reported here in this appendix.

The data can be broken down by looking at results of individual problems. In both tests given in appendices F and G, problems 1 and 3 are mathematically isomorphic to each other, their only difference being a reversing of the referents of the unknowns. For example, in problem 1, the price of turtles was given, the number of turtles ¬as unknown. In problem 3, that is reversed. Similarly, problems 2 and 4 are mathematically isomorphic. One of the goals of the diagnostic test (which was tangential to the present study) was to test the effect of switching the referents in the above manner.

Problem No.	No. of students attempting the problem	No. of correct solutions	%correct
1	54	34	63%
2	47	16	34%
3	47	30	64%
4	54	22	41%

Table 10: Results by problem of the first written test.

Table 10 gives the percentage of problems done correctly in the first test 1. Theses data imply that there is vitually no difference between the results of problem 1 and 3. The difference between problems 2 and 4 is also small. However, the difference between problems 1 and 3 combined ("turtles" problems) as compared with problems 2 and 4 combined ("blocks" problems) is a significant one. 64% of the "turtles" problems were done correctly as compared with 38% of the "blocks" problems. These data are further broken down in Table 11 showing the number of students who got both, neither, or one or the other problem correct. McNemar's test was used to test the hypothesis that the probability that a student getting the "turtles" problem correct and the "blocks" problem wrong is equal to the probability of a student getting the "blocks" problem correct and the "turtles" problem wrong.

	Turtles problem correct	Turtles problem wrong	Totals
Blocks problem correct	34	4	38
Blocks problem wrong	30	33	63
Totals	64	37	101

Table 11: Comparison of results of "turtles" and "blocks" problems in first test.

Using a chi squared approximation, the hypothesis can be rejected (with a significance level, $p<.001$) thus impling that the "turtles" problem was significantly easier than the "blocks" problem.

One plausible explaination for such a difference was the difference in the complexity of the numbers. The turtles problems via their most common solutions require that 60 be divided by 15 to get answers of 4 and ultimately 12. In the blocks problem, 91 is divided by 7 to get 13 and ultimately 52.

This motivated the administering of test 2 which was identical to test 1 in all aspects except the numbers used were switched between the "turtles" and "blocks" problems. If the size of the numbers does play an important role in the success or failure in these types of problems, one would expect a significant difference between the "blocks" and "turtles" problems in the opposite direction, blocks problems being the easier ones. This did not occur.

Table 12 gives the results of test 2 broken down by problem and Table 13 is analogous to Table 11.

Problem No.	No. of students attempting the problem	No. of correct solutions	%correct
1	76	32	41%
2	77	28	38%
3	77	44	57%
4	76	40	53%

TABLE 12: Results by problem of the second written test.

	Turtles problem correct	Turtles problem wrong	Totals
Blocks problem correct	44	24	69
Blocks problem wrong	32	53	84
Totals	75	78	153

TABLE 13: Comparison of results of "turtles" and "blocks" problems in second test.

McNemar's Test, used on the data in Table 13, failed to disclose a significant difference between success on the "turtles" problems and success on the "blocks" problems. What little numerical difference there is seems to point in the direction of the turtles problem still being the easier one, contrary to what had been expected.

The statistic that 49% of the students solved "turtles" problems correctly in Test 2, however, is significantly different ($p<.02$) from the percent of students who got the "turtles" problems correct in Test 1 (64%). This might indicate that the size of the numbers has some effect on success or failure in solving these algebra word problems. It seems clear, however, that the difference in the sizes of the numbers alone does not adequately explain the difference in success rates in Test 1 between "turtles" and "blocks" problems.

It is interesting to note in Table 12 that there does seem to be a difference between problems 1 and 4 and problems 2 and 3. In Test 2, unlike Test 1, it seems that problems where quantity was left unknown were more difficult than problems where weight or price were unknown. Table 14 (on the following page) is similar to Table 13 except that the former compares problems where the quantity is unknown with problems where price or weight is unknown, whereas the latter compares "turtles" problems with "blocks" problems.

	Problem 3 or 4 correct	Problem 3 or 4 wrong	Totals
Problem 1 or 2 (Quantity unknown) correct	44	16	60
Problem 1 or 2 (Quantity unknown) wrong	40	53	93
Totals	84	69	153

TABLE 14: Comparison of results of problems 1 and 2 with those of problems 3 and 4 in second writteen test.

Using McNemar's Test, the hypothesis that a student is just as likely to get problems 1 or 2 correct and problems 3 or 4 wrong as the reverse, was rejected (with a significance level of $p<.025$) implying that problems 1 and 2 are more difficult than problems 3 and 4.

These results , when compared with the first test seem anomalous. More research is required to further understand what is going on.